# Ceres in Intuitionistic Logic

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**Abstract**

In this paper we present a procedure allowing the extension of a `CERES`-based cut-elimination method to intuitionistic logic. Previous results concerning this problem manage to capture fragments of intuitionistic logic, but in many essential cases structural constraints were violated during normal form construction resulting in a classical proof. The heart of the `CERES` method is the resolution calculus, which ignores the structural constraints of the well known intuitionistic sequent calculi. We propose, as a method of avoiding the structural violations, the generalization of resolution from the resolving of clauses to the resolving of cut-free proofs, in other words, what we call *proof resolution*. The result of proof resolution is a cut-free proof rather than a clause. Note that, resolution on ground clauses is essentially atomic cut, thus using proof resolution to construct cut-free proofs one would need to *join* the two proofs together and remove the atoms which where resolved. To efficiently perform this joining (and guarantee that the resulting cut-free proof is intuitionistic) we develop the concept of *proof subsumption* (similar to clause subsumption) and *indexed resolution*, a refinement indexing atoms by their corresponding positions in the cut formula. Proof subsumption serves as a tool to prove the completeness of the new method `CERES`-i, and indexed resolution provides an efficient strategy for the joining of two proofs which is in general a non-deterministic search. Such a refinement is essential for any attempt to implement this method. Finally we compare the complexity of `CERES`-i with that of Gentzen-based methods.

## 1. Introduction

Cut-elimination was originally introduced by Gerhard Gentzen as a theoretical tool from which results like decidability and consistency could be proven [10]. Cut-free proofs are computationally explicit objects from which interesting information such as Herbrand disjunctions and interpolants can easily be extracted.

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When viewing formal proofs as a model for mathematical proofs, cut-elimination corresponds to the removal of lemmas, which leads to interesting applications (see, e.g. [2, 3]).

For such applications to mathematical proofs, the cut-elimination method CERES (cut-elimination by resolution) was developed in [5]; initially, the method was developed for classical first-order logic. It essentially reduces cut-elimination for a proof $\varphi$ of a sequent $S$ to a theorem proving problem: the refutation of a clause set corresponding to $\varphi$, denoted by $\mathrm{CL}(\varphi)$. Given a resolution refutation of $\mathrm{CL}(\varphi)$, an essentially cut-free proof (a proof with only atomic cuts) can be constructed by a proof-theoretic transformation. This proof theoretic transformation uses so-called proof projections $\varphi[C]$ for $C \in \mathrm{CL}(\varphi)$, which are simple cut-free proofs extracted from $\varphi$ (proving the end-sequent $S$ extended by the atomic sequent $C$).

Due to the reduction to a theorem proving problem encoding crucial structural properties of cut, CERES turned out to be a powerful tool in proof analysis [3]. Moreover, its asymptotic complexity is superior to that of two reductive Gentzen-style methods [7]. The original method CERES was designed for classical first-order logic. Extensions to higher-order logic and first-order proof schemata were defined in [12] and [9], respectively. As intuitionistic proofs, like classical ones, are natural formalisms for mathematical reasoning, they are of major importance to proof mining (see e.g. [13]). Therefore, it is a natural question whether the method CERES can be extended to intuitionistic logic. However, the naive extension of CERES to first-order intuitionistic logic does not work, as the results of the CERES-transformations are classical proofs in general. In [16], it was shown that, for some intuitionistic proofs, there are refutations of the clause set which cannot be transformed into intuitionistic proofs: neither are the CERES-normal forms intuitionistic nor can they be transformed into intuitionistic proofs by reasonable proof transformations. Only for proofs $\varphi$ of sequents of the form $\Gamma \vdash$, the CERES-method can be maintained, provided negative resolution refutations are applied to $\mathrm{CL}(\varphi)$ [16]. This suggests that, in order to cover all intuitionistic proofs, a radical change of the CERES-method is required.

In this paper, we develop a complete CERES-like method for intuitionistic proofs $\varphi$ with skolemized end-sequents, called CERES-i. In Section 3.1, we show that a separation of projections and resolution refutations (which is characteristic to first-order CERES) does not work for intuitionistic logic; in fact there are proofs $\varphi$ of a sequent $S$ and resolution refutations of $\mathrm{CL}(\varphi)$ which cannot be combined with the projections to construct an intuitionistic cut-free proof of $S$. Our solution of this problem consists in generalizing the resolution calculus from clauses to cut-free proofs: instead of resolving clauses $C, D \in \mathrm{CL}(\varphi)$, we resolve their projections $\varphi[C]$ and $\varphi[D]$, resulting in a new cut-free proof. We introduce this general resolution principle of proofs in Section 6. The completeness of proof resolution in intuitionistic logic (the derivation yields a cut-free intuitionistic proof) is based on a subsumption principle for proofs which is defined in Section 5. These results yield a method, called CERES-i (defined in Section 8),

$$
\begin{tikzcd}
 & & \varphi \arrow[dl] \arrow[ddr] & \\
 & \varphi^t \arrow[dr] \arrow[ddl] & & \\
 & & \mathcal{P}(\varphi^t) \arrow[d] & \mathcal{P}(\varphi) \arrow[l, dashed, "\text{subsumes}"'] \arrow[d] \\
\begin{array}{c}\text{Elimination of}\\ \text{atomic cuts}\end{array} \arrow[d] & & \begin{array}{c}\text{Proof}\\ \text{resolution}\end{array} \arrow[ll, dashed, "\text{subsumes}"'] \arrow[d] & \begin{array}{c}\text{Proof}\\ \text{resolution}\end{array} \arrow[l, dashed, "\text{subsumes}"'] \arrow[d] \\
\begin{array}{c}\text{Cut-free}\\ \textbf{LJ}\text{ proof}\end{array} & & \begin{array}{c}\text{Cut-free}\\ \text{proof}\end{array} \arrow[ll, dashed, "\text{subsumes}"'] & \begin{array}{c}\text{Cut-free}\\ \text{proof}\end{array} \arrow[l, dashed, "\text{subsumes}"']
\end{tikzcd}
$$

Figure 1: Overview of the results.

for cut-elimination in intuitionistic logic: given an intuitionistic proof $\varphi$ of a sequent $S$, we first compute the set of projections $\mathcal{P}(\varphi)$ of $\varphi$; then we apply proof resolution to $\mathcal{P}(\varphi)$ and derive a cut-free intuitionistic proof $\psi$ of $S$.

The results are better summarized in the diagram of Figure 1, where $\varphi$ represents an **LJ** proof with cuts, $\varphi^t$ contains only atomic cuts on axioms (obtained from $\varphi$ via reductive cut-elimination), and $\mathcal{P}(\psi)$ denotes the set of projections of a proof $\psi$. The rightmost branch represents the `CERES`-i method proposed, and the leftmost branch is a specific reductive cut-elimination strategy. The middle branch serves as a bridge to show the completeness of `CERES`-i, i.e., that the final cut-free proof obtained is intuitionistic. This is done via the proof subsumption property, which is indicated in the diagram by the horizontal edges.

In Section 8, we also define a complete refinement of `CERES`-i which reduces proof search. In Section 10, we compare `CERES`-i with a refinement of the reductive cut-elimination method shown on the leftmost branch of Figure 1 and show that `CERES`-i asymptotically outperforms the reductive method.

In summary, we define a novel method for cut-elimination in intuitionistic logic which unifies methods from proof theory and the resolution calculus. We demonstrate that principles like resolution and subsumption, which are powerful tools in automated deduction, can be generalized to cut-free proofs. We think that this methodology might be fruitful in investigating the complexity of cut-elimination and in comparing different cut-elimination methods. Of course, all the results in this paper also hold for classical logic. Generally, the methods of subsumption and proof resolution may provide useful tools for proof analysis in a more general context.

## 2. Preliminaries

We assume that the reader is familiar with the syntax of first-order logic, sequent calculus and resolution.

A sequent is a structure $\Gamma \vdash \Delta$, where $\Gamma$ and $\Delta$ are *multisets* of formulas. We will use the multiplicative versions of the calculi **LK** (Figure 2) and **LJ** (Figure 3). Note that the **LK** rule for $\vee$-left is not the usual one, but it will be necessary for the transformation we define later on.

Given a proof $\varphi$, we use the following notations:

- $\mathrm{es}(\varphi)$ denotes the end-sequent of $\varphi$.
- $\varphi.\nu$ is the subproof of $\varphi$ rooted at node $\nu$.
- $\varphi[\chi]_\nu$ is the replacement of the subproof at $\varphi.\nu$ by the proof $\chi$.

We define the *ancestors* of a formula $F$ in a proof $\varphi$ inductively as follows:

– Let $F \in es(\varphi.\nu)$ then $F$ is an ancestor of $F$.

– If $\varphi.\mu$ is a subproof of $\varphi.\nu$, $\rho$ is the last inference of $\varphi.\mu$, and $F' \in es(\varphi.\mu)$ is an ancestor of $F$, then one of the following cases must hold:

  - If $\rho$ is an axiom then we are done
  - If $\rho$ is a unary inference with $F'$ as the main formula, then the auxiliary formula of $\rho$ is an ancestor of $F$.
  - If $\rho$ is a unary inference where $F'$ is not the main formula, then the ancestor set remains the same.
  - If $\rho$ is a binary inference with $F'$ as the main formula, then the auxiliary formulas of $\rho$ are ancestors of $F$.
  - If $\rho$ is a binary inference where $F'$ is not the main formula, then the ancestor set remains the same.

In some situations, we will need to reason about the free variables occurring in a formula, set of formulas, or proof. We denote this set of variables by $V(e)$, where $e$ might be any of the elements previously mentioned.

In a sequent, universal quantifiers of negative polarity and existential quantifiers of positive polarity are called *weak*, universal quantifiers of positive polarity and existential quantifiers of negative polarity are called *strong*. A sequent $S$ is called *skolemized* if only weak quantifiers occur in $S$. A proof of $S$ is called *skolemized* if $S$ is skolemized [4]. Skolemization for the calculus **LJ** can be accomplished using epsilon terms [17]. We postpone the discussion on how epsilonization can be used for `CERES`-i to Section 8.1, after the method is defined.

The inferences $\forall_l$ and $\exists_r$ are called *weak*, $\forall_r$, $\exists_l$ are called *strong*. Note that a cut-free proof of a skolemized sequent does not contain strong quantifiers; skolemized proofs with cuts may contain strong quantifier inferences but their principal formulas are ancestors of cut formulas.

$$\frac{}{A \vdash A}\ init \qquad \frac{\Gamma_1 \vdash \Delta_1, P \quad \Gamma_2, P \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ cut$$

$$\frac{\Gamma \vdash \Delta, P}{\Gamma, \neg P \vdash \Delta}\ \neg : l \qquad \frac{\Gamma, P \vdash \Delta}{\Gamma \vdash \Delta, \neg P}\ \neg : r$$

$$\frac{P_i, \Gamma \vdash \Delta}{P_1 \wedge P_2, \Gamma \vdash \Delta}\ \wedge : l_i \qquad \frac{\Gamma_1 \vdash \Delta_1, P \quad \Gamma_2 \vdash \Delta_2, Q}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, P \wedge Q}\ \wedge : r$$

$$\frac{P, \Gamma_1 \vdash \Delta, \Delta_1 \quad Q, \Gamma_2 \vdash \Delta, \Delta_2}{P \vee Q, \Gamma_1, \Gamma_2 \vdash \Delta, \Delta_1, \Delta_2}\ \vee : l \qquad \frac{\Gamma \vdash \Delta, P_i}{\Gamma \vdash \Delta, P_1 \vee P_2}\ \vee : r_i$$

$$\frac{\Gamma_1 \vdash \Delta_1, P \quad Q, \Gamma_2 \vdash \Delta_2}{P \rightarrow Q, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \rightarrow : l \qquad \frac{\Gamma, P \vdash \Delta, Q}{\Gamma \vdash \Delta, P \rightarrow Q}\ \rightarrow : r$$

$$\frac{P\{x \leftarrow \alpha\}, \Gamma \vdash \Delta}{\exists x.P, \Gamma \vdash \Delta}\ \exists : l \qquad \frac{\Gamma \vdash \Delta, P\{x \leftarrow t\}}{\Gamma \vdash \Delta, \exists x.P}\ \exists : r$$

$$\frac{P\{x \leftarrow t\}, \Gamma \vdash \Delta}{\forall x.P, \Gamma \vdash \Delta}\ \forall : l \qquad \frac{\Gamma \vdash \Delta, P\{x \leftarrow \alpha\}}{\Gamma \vdash \Delta, \forall x.P}\ \forall : r$$

$$\frac{P, P, \Gamma \vdash \Delta}{P, \Gamma \vdash \Delta}\ c : l \qquad \frac{\Gamma \vdash \Delta, P, P}{\Gamma \vdash \Delta, P}\ c : r$$

$$\frac{\Gamma \vdash \Delta}{P, \Gamma \vdash \Delta}\ w : l \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, P}\ w : r$$

Figure 2: **LK**: Sequent calculus for classical logic. It is assumed that $\alpha$ is a variable not contained in $P$, $\Gamma$ or $\Delta$, $t$ does not contain variables bound in $P$ and $i \in \{1, 2\}$, $A$ is an atomic formula.

$$\frac{}{A \vdash A}\ init \qquad \frac{\Gamma_1 \vdash P \quad \Gamma_2, P \vdash C}{\Gamma_1, \Gamma_2 \vdash C}\ cut$$

$$\frac{\Gamma \vdash P}{\Gamma, \neg P \vdash}\ \neg : l \qquad \frac{\Gamma, P \vdash}{\Gamma \vdash \neg P}\ \neg : r$$

$$\frac{P_i, \Gamma \vdash C}{P_1 \wedge P_2, \Gamma \vdash C}\ \wedge : l_i \qquad \frac{\Gamma_1 \vdash P \quad \Gamma_2 \vdash Q}{\Gamma_1, \Gamma_2 \vdash P \wedge Q}\ \wedge : r$$

$$\frac{P, \Gamma_1 \vdash C \quad Q, \Gamma_2 \vdash C}{P \vee Q, \Gamma_1, \Gamma_2 \vdash C}\ \vee : l \qquad \frac{\Gamma \vdash P_i}{\Gamma \vdash P_1 \vee P_2}\ \vee : r_i$$

$$\frac{\Gamma_1 \vdash P \quad Q, \Gamma_2 \vdash C}{P \rightarrow Q, \Gamma_1, \Gamma_2 \vdash C}\ \rightarrow : l \qquad \frac{\Gamma, P \vdash Q}{\Gamma \vdash P \rightarrow Q}\ \rightarrow : r$$

$$\frac{P\{x \leftarrow \alpha\}, \Gamma \vdash C}{\exists x.P, \Gamma \vdash C}\ \exists : l \qquad \frac{\Gamma \vdash P\{x \leftarrow t\}}{\Gamma \vdash \exists x.P}\ \exists : r$$

$$\frac{P\{x \leftarrow t\}, \Gamma \vdash C}{\forall x.P, \Gamma \vdash C}\ \forall : l \qquad \frac{\Gamma \vdash P\{x \leftarrow \alpha\}}{\Gamma \vdash \forall x.P}\ \forall : r$$

$$\frac{P, P, \Gamma \vdash C}{P, \Gamma \vdash C}\ c : l \qquad \frac{\Gamma \vdash C}{P, \Gamma \vdash C}\ w : l \qquad \frac{\Gamma \vdash}{\Gamma \vdash P}\ w : r$$

Figure 3: **LJ**: Sequent calculus for intuitionistic logic. It is assumed that $\alpha$ is a variable not contained in $P$, $\Gamma$ or $C$ and $t$ does not contain variables bound in $P$, $A$ is an atomic formula. Also, $C$ stands for one formula or the empty multiset.

## 3. Cut-elimination by resolution

Since our method is based on the original `CERES`, we will briefly present it here. The method `CERES` can be sketched as follows. Let $\varphi$ be a (skolemized) proof with cuts of a closed sequent $S$. We start by collecting the atoms that occur in the axioms and are ancestors of cut-formulas. These atoms are organized in clauses, which form the *characteristic clause set*. This organization depends on the side in which they occur in the axiom and from which branch they come from on binary rules. Thus, each clause in this set will be composed of a number of atoms which came from cut-formulas and occur in some axiom of $\varphi$. The next step consists of constructing the so-called *projections*, one for each of the clauses in the characteristic clause set. A projection for a clause $C$ will be constructed using the inferences of $\varphi$. We start from the axioms containing occurrences of the atoms of $C$ and continue until we reach the end sequent $S$, skipping cuts and inferences on cut ancestors. Since not all axioms are available, the weakening of some formulas might be necessary in the projection construction. Thus, projections are *cut-free* derivations of $S$ plus the atoms from $C$. Concerning the characteristic clause set, a *resolution refutation* of it ought to be found (it has been shown that this set is always refutable). The refutation results in a derivation of the empty clause (also represented by the empty sequent $\vdash$) which will be used as the skeleton for the new proof. The new proof will have only atomic cuts, which correspond to the applications of the resolution rule in the refutation. It is assembled using the *context product* of $S$ and the grounded resolution refutation. This product is a derivation of $S$ from sequents composed of $S$ and atoms from $C$. Then the projections are used as proofs of this derivation's open premises.

We will now formally define all these elements and operations, most of which will be used throughout the paper. At the same time, we will apply the method to the following **LK** proof, where the cut-ancestors are **bold**:

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\dfrac{\dfrac{}{\mathbf{P}\alpha \vdash P\alpha}}{\neg P\alpha, \mathbf{P}\alpha \vdash}\,\neg_l
}{\neg P\alpha \vdash \neg \mathbf{P}\alpha}\,\neg_r
\qquad
\dfrac{}{Q\alpha \vdash \mathbf{Q}\alpha}
}{\neg P\alpha \vee Q\alpha \vdash \neg \mathbf{P}\alpha, \mathbf{Q}\alpha}\,\vee_l
}{\neg P\alpha \vee Q\alpha \vdash \neg \mathbf{P}\alpha \vee \mathbf{Q}\alpha, \neg \mathbf{P}\alpha \vee \mathbf{Q}\alpha}\,\vee_r \times 2
}{\neg P\alpha \vee Q\alpha \vdash \neg \mathbf{P}\alpha \vee \mathbf{Q}\alpha}\,c_r
}{\forall x.(\neg Px \vee Qx) \vdash \neg \mathbf{P}\alpha \vee \mathbf{Q}\alpha}\,\forall_l
}{\forall x.(\neg Px \vee Qx) \vdash \exists \mathbf{y}.(\neg \mathbf{P}\alpha \vee \mathbf{Q}\mathbf{y})}\,\exists_r
}{\forall x.(\neg Px \vee Qx) \vdash \forall \mathbf{x}.\exists \mathbf{y}.(\neg \mathbf{P}\mathbf{x} \vee \mathbf{Q}\mathbf{y})}\,\forall_r
\qquad
\dfrac{
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\dfrac{\dfrac{}{Pa \vdash \mathbf{Pa}}}{\neg \mathbf{Pa}, Pa \vdash}\,\neg_l
}{\neg \mathbf{Pa} \vdash \neg Pa}\,\neg_r
\qquad
\dfrac{}{\mathbf{Q}\beta \vdash Q\beta}
}{\neg \mathbf{Pa} \vee \mathbf{Q}\beta \vdash \neg Pa, Q\beta}\,\vee_l
}{\neg \mathbf{Pa} \vee \mathbf{Q}\beta \vdash \neg Pa \vee Q\beta, \neg Pa \vee Q\beta}\,\vee_r \times 2
}{\neg \mathbf{Pa} \vee \mathbf{Q}\beta \vdash \neg Pa \vee Q\beta}\,c_r
}{\neg \mathbf{Pa} \vee \mathbf{Q}\beta \vdash \exists x.\exists y.(\neg Px \vee Qy)}\,\exists_r \times 2
}{\exists \mathbf{y}.(\neg \mathbf{Pa} \vee \mathbf{Q}\mathbf{y}) \vdash \exists x.\exists y.(\neg Px \vee Qy)}\,\exists_l
}{\forall \mathbf{x}.\exists \mathbf{y}.(\neg \mathbf{P}\mathbf{x} \vee \mathbf{Q}\mathbf{y}) \vdash \exists x.\exists y.(\neg Px \vee Qy)}\,\forall_l
}{\forall x.(\neg Px \vee Qx) \vdash \exists x.\exists y.(\neg Px \vee Qy)}\,\mathit{cut}
$$

**Definition 3.1 (Characteristic clause set).** Let $\varphi$ be a skolemized proof. The characteristic clause set is built recursively from the leaves of the proof until the end sequent. Let $\nu$ be an occurrence of a sequent in this proof. Then:

- If $\nu$ is an axiom, then $\mathrm{CL}(\nu)$ contains the sub-sequent of $\nu$ composed only of cut-ancestors.
- If $\nu$ is the result of the application of a unary rule on a sequent $\mu$, then $\mathrm{CL}(\nu) = \mathrm{CL}(\mu)$
- If $\nu$ is the result of the application of a binary rule on sequents $\mu_1$ and $\mu_2$, then we distinguish two cases:
  - If the rule is applied to ancestors of any cut-formula, then $\mathrm{CL}(\nu) = \mathrm{CL}(\mu_1) \cup \mathrm{CL}(\mu_2)$
  - If the rule is applied to ancestors of the end-sequent, then $\mathrm{CL}(\nu) = \mathrm{CL}(\mu_1) \times \mathrm{CL}(\mu_2)$

Where[1]: $\mathrm{CL}(\mu_1) \times \mathrm{CL}(\mu_2) = \{C \circ D \mid C \in \mathrm{CL}(\mu_1), D \in \mathrm{CL}(\mu_2)\}$.

If $\nu_0$ is the root node, $\mathrm{CL}(\nu_0)$ is called the characteristic clause set of $\varphi$, also denoted by $\mathrm{CL}(\varphi)$.

The characteristic clause set of our example proof is:

$$(\{P\alpha \vdash\} \times \{\vdash Q\alpha\}) \cup (\{\vdash Pa\} \cup \{Q\beta \vdash\})$$

Which reduces to:

$$\mathrm{CL}(\varphi) = \{P\alpha \vdash Q\alpha \ ; \ \vdash Pa \ ; \ Q\beta \vdash\}$$

**Definition 3.2 (Projection).** Let $\varphi$ be a proof and $\nu$ a node which is a conclusion of an inference $\xi$. We define $\mathcal{P}(\varphi.\nu)$ as the set of projections $\{\varphi.\nu[C] \mid C \in \mathrm{CL}(\varphi.\nu)\}$. Each projection $\varphi.\nu[C]$ is a cut-free proof of the sequent $S(\nu) \circ C$[2] inductively defined as follows:

- If $\xi$ is an axiom, then $\mathcal{P}(\varphi.\nu) = \{\varphi\}$.
- If $\xi$ is a unary rule with premise $\mu$:
  - If $\xi$ operates on a cut-ancestor, then $\mathcal{P}(\varphi.\nu) = \mathcal{P}(\varphi.\mu)$.
  - If $\xi$ operates on an end-sequent ancestor, then $\mathcal{P}(\varphi.\nu)$ is the set of:

    $$\frac{\varphi.\mu[C_i]}{\zeta}\ \xi$$

    such that $\varphi.\mu[C_i] \in \mathcal{P}(\varphi.\mu)$.
- If $\xi$ is a binary rule with premises $\mu_1$ and $\mu_2$:
  - If $\xi$ operates on a cut-ancestor, then $\mathcal{P}(\varphi.\nu) = \mathcal{P}(\varphi.\mu_1) \cup \mathcal{P}(\varphi.\mu_2)$.

[1] $(\Gamma \vdash \Delta) \circ (\Gamma' \vdash \Delta') = \Gamma, \Gamma' \vdash \Delta, \Delta'$.

[2] $S(\nu)$ is the sequent occurring at node $\nu$

– If $\xi$ operates on an end-sequent ancestor, then $\mathcal{P}(\varphi.\nu)$ is the set of:

$$\dfrac{\varphi.\mu_1[C_i^1] \quad \varphi.\mu_2[C_j^2]}{\zeta}\ \xi$$

such that $\varphi.\mu_1[C_i^1] \in \mathcal{P}(\varphi.\mu_1)$ and $\varphi.\mu_2[C_j^2] \in \mathcal{P}(\varphi.\mu_2)$.

- For any $C \in \mathrm{CL}(\varphi)$ we define $\varphi[C] = \varphi.\nu_0[C]$ where $\nu_0$ is the root node of $\varphi$.

In each step, it might be necessary to weaken the auxiliary formulas of an inference. Moreover, if not all formulas of the end-sequent are present after constructing the projection, they may be weakened as well.

Note that no rule operates on cut ancestors, therefore they occur as atoms in the end-sequent of the projections.

For a proof $\varphi$ we define the set of projections: $\mathcal{P}(\varphi) = \{\varphi[C] \mid C \in \mathrm{CL}(\varphi)\}$.

For constructing a projection of a proof $\varphi$ of an end-sequent $S$ and a $C \in \mathrm{CL}(\varphi)$ it is in general not necessary to construct a proof of $C \circ S$. Instead we may construct a proof of $C \circ S'$ where $S'$ is a subsequent of $S$ by minimizing the weakenings on ancestors of the end-sequent. When we have such a projection $\varphi[C]$ and further omission of weakenings to reduce the size of $S'$ are impossible we speak about a *minimal projection*. Both *full projections* (proofs $\varphi[C]$ of $C \circ S$) and minimal projections (proofs $\varphi[C]$ of $C \circ S'$) will be called projections.

When defining the projections, note that binary rules operating on cut-ancestors will not combine the derivations but only take their union. This will result in some projections coming from the left branch and others coming from the right branch of these rules. If $\varphi_1$ and $\varphi_2$ are two subproofs corresponding to different branches, we say that a projection either *goes over* $\varphi_1$ or $\varphi_2$.

These are the (full) projections for the clauses obtained before:

$\varphi[\vdash Pa]$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Pa \vdash \mathbf{Pa}}}{\vdash \mathbf{Pa}, \neg Pa}\ \neg_r}{\vdash \mathbf{Pa}, \neg Pa, Q\beta}\ w_r}{\vdash \mathbf{Pa}, \neg Pa \lor Q\beta, \neg Pa \lor Q\beta}\ \lor_r \times 2}{\vdash \mathbf{Pa}, \neg Pa \lor Q\beta}\ c_r}{\vdash \mathbf{Pa}, \exists x.\exists y.(\neg Px \lor Qy)}\ \exists_r \times 2}{\forall x.(\neg Px \lor Qx) \vdash \mathbf{Pa}, \exists x.\exists y.(\neg Px \lor Qy)}\ w_l$$

$\varphi[Q\beta \vdash]$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{Q}\beta \vdash Q\beta}}{\mathbf{Q}\beta \vdash Q\beta, \neg Pa}\ w_r}{\mathbf{Q}\beta \vdash \neg Pa \lor Q\beta, \neg Pa \lor Q\beta}\ \lor_r \times 2}{\mathbf{Q}\beta \vdash \neg Pa \lor Q\beta}\ c_r}{\mathbf{Q}\beta \vdash \exists x.\exists y.(\neg Px \lor Qy)}\ \exists_r \times 2}{\forall x.(\neg Px \lor Qx), \mathbf{Q}\beta \vdash \exists x.\exists y.(\neg Px \lor Qy)}\ w_l$$

$\varphi[P\alpha \vdash Q\alpha]$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\mathbf{P}\alpha \vdash P\alpha}}{\neg P\alpha, \mathbf{P}\alpha \vdash} \neg_l \quad \overline{Q\alpha \vdash \mathbf{Q}\alpha}}{\neg P\alpha \vee Q\alpha, \mathbf{P}\alpha \vdash \mathbf{Q}\alpha} \vee_l}{\forall x.(\neg Px \vee Qx), \mathbf{P}\alpha \vdash \mathbf{Q}\alpha} \forall_l}{\forall x.(\neg Px \vee Qx), \mathbf{P}\alpha \vdash \mathbf{Q}\alpha, \exists x.\exists y.(\neg Px \vee Qy)} w_r$$

The minimal projections can be obtained by omitting the last weakenings of the projections shown above.

**Definition 3.3 (Resolution calculus).** The resolution calculus is composed of the following inference rules:

$$\dfrac{\Gamma \vdash \Delta, A \quad \Gamma', A' \vdash \Delta'}{\Gamma\sigma, \Gamma'\sigma \vdash \Delta\sigma, \Delta'\sigma} R \quad \dfrac{\Gamma, A, A' \vdash \Delta}{\Gamma\sigma, A\sigma \vdash \Delta\sigma} C_l \quad \dfrac{\Gamma \vdash \Delta, A, A'}{\Gamma\sigma \vdash \Delta\sigma, A\sigma} C_r$$

Where $\sigma$ is the most general unifier (m.g.u.) of $A$ and $A'$.

Here we show a resolution refutation and its grounded version for the clause set of our example:

$$\dfrac{\dfrac{\vdash Pa \quad P\alpha \vdash Q\alpha}{\vdash Qa} R\{\alpha \leftarrow a\} \quad Q\beta \vdash}{\vdash} R\{\beta \leftarrow a\} \qquad \dfrac{\dfrac{\vdash Pa \quad Pa \vdash Qa}{\vdash Qa} R \quad Qa \vdash}{\vdash} R$$

Notice how the resolution rule looks like an atomic cut, except for the unifier. If the resolution is ground, then it is exactly an atomic cut.

**Definition 3.4 (Context product).** Let $C$ be a sequent and $\varphi$ an **LK** derivation with end-sequent $S$ such that no free variable in $C$ occurs as an eigenvariable in $\varphi$. We define the *context product* $C \star \varphi$ (which gives a derivation of $C \circ S$) inductively:

- If $\varphi$ consists only of an axiom, then $C \star \varphi$ is the sequent: $C \circ S$.
- If $\varphi$ ends with a unary rule $\xi$:

$$\dfrac{\begin{array}{c}\varphi' \\ S'\end{array}}{S} \xi$$

  then we assume that $C \star \varphi'$ is already defined and thus $C \star \varphi$ is:

$$\dfrac{\begin{array}{c}C \star \varphi' \\ C \circ S'\end{array}}{C \circ S} \xi$$

  Since $C$ does not contain free variables which are eigenvariables of $\varphi$, the context product is well-defined even in the cases of $\xi \in \{\forall_r, \exists_l\}$, although this case does not occur in our setting.

- If $\varphi$ ends with a binary rule $\xi$:

$$\dfrac{\begin{matrix}\varphi_1 & \varphi_2\\ S_1 & S_2\end{matrix}}{S}\ \xi$$

then assume that $C\star\varphi_1$ and $C\star\varphi_2$ are already defined. We define $C\star\varphi$:

$$\dfrac{\dfrac{\begin{matrix}C\star\varphi_1 & C\star\varphi_2\\ C\circ S_1 & C\circ S_2\end{matrix}}{C\circ C\circ S}\ \xi}{C\circ S}\ c^*$$

Note that since the formulas in $C$ will come from both branches, and we are working in a multiplicative calculus, after applying a binary rule we need to contract the formulas from $C$ to obtain the correct multiset.

Let $F = \forall x.(\neg Px \lor Qx)$ and $G = \exists x.\exists y.(\neg Px \lor Qy)$. The context product of $F \vdash G$ (the end-sequent of our example proof) and the ground resolution refutation above is:

$$\dfrac{\dfrac{\dfrac{\dfrac{F\vdash G, Pa \quad Pa, F\vdash G, Qa}{F, F\vdash G, G, Qa}\ cut}{F\vdash G, Qa}\ c_l, c_r \qquad Qa, F\vdash G}{F, F\vdash G, G}\ cut}{F\vdash G}\ c_l, c_r$$

Notice that the open premises of this derivation are exactly the end-sequents of the projections, given the proper instantiations.

**Definition 3.5 (CERES).** Let $\varphi$ be an **LK** proof of a skolemized sequent $S$, $\mathrm{CL}(\varphi)$ its clause set and $\varrho$ a grounded resolution refutation of $\mathrm{CL}(\varphi)$. We first construct $\varrho' = S\star\varrho$. Note that this is a derivation of $S$ from a set of axioms $C\circ S$, with $C \in \mathrm{CL}(\varphi)$, which are exactly the end-sequents of the projections $\mathcal{P}(\varphi)$. Now we define $\varphi(\varrho)$ by replacing all axioms of $\varrho'$ by the respective projections. By definition, $\varphi(\varrho)$ is an **LK** proof of $S$ with only atomic cuts. We call it a CERES *normal form* of $\varphi$ with respect to $\varrho$.

As a final step, the context product is joined with the instantiated projections in a proof with only atomic cuts (Note that the projections have been abbreviated):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Pa\vdash \mathbf{Pa}}}{\vdash \mathbf{Pa}, \neg Pa}\ \neg_r}{\vdash \mathbf{Pa}, \neg Pa\lor Q\beta}\ \lor_r}{\vdash \mathbf{Pa}, \exists x.\exists y.(\neg Px\lor Qy)}\ \exists^*_r}{F\vdash \mathbf{Pa}, G}\ w_l \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{Pa}\vdash Pa}}{\neg Pa, \mathbf{Pa}\vdash}\ \neg_l \quad \dfrac{}{Qa\vdash \mathbf{Qa}}}{\neg Pa\lor Qa, \mathbf{Pa}\vdash \mathbf{Qa}}\ \lor_l}{\forall x.(\neg Px\lor Qx), \mathbf{Pa}\vdash \mathbf{Qa}}\ \forall_l}{F, \mathbf{Pa}\vdash \mathbf{Qa}, G}\ w_r}{F, F\vdash G, G, Qa}\ cut}{F\vdash G, Qa}\ c_l, c_r \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\mathbf{Qa}\vdash Qa}}{\mathbf{Qa}\vdash \neg Pa\lor Qa}\ \lor_r}{\mathbf{Qa}\vdash \exists x.\exists y.(\neg Px\lor Qy)}\ \exists^*_r}{F, \mathbf{Qa}\vdash G}\ w_l}{F, F\vdash G, G}\ cut}{F\vdash G}\ c_l, c_r$$

**Remark 3.1.** *A* `CERES` *normal form can also be defined via minimal projections; in this case fewer contractions are needed to obtain the end-sequent. Additional weakenings might be necessary to obtain the full end-sequent (though not in the example above).*

*3.1. Extending to Intuitionistic Logic*

If we try to apply `CERES` to an **LJ** proof, we encounter at first sight a few technical problems. Although it is possible to obtain a characteristic clause set and a resolution refutation, the projections and context product will, in general, be classical derivations. The occurrence of positive atoms in clauses, i.e. atoms on the right side of $\vdash$, will possibly be in conflict with end-sequent ancestors on the right, and **LJ** is a single conclusion calculus.

We have tried to work around this problem in a number of (simpler) ways, but only to realize it worked for a fragment of **LJ**. For example, in [15], we proposed a new resolution calculus that contains single-conclusion negation rules and resolution on negated atoms. The clause set can thus be modified to have no atoms on the right side, by using their negations on the left. Still violations of **LJ**'s single-conclusion restriction might occur by negated atoms on the right. Other attempts involved eliminating the atomic cuts *a posteriori*, either by using Gentzen's original rewriting rules or modified versions of them [16, Chapters 7 & 8]. Multi-conclusion calculi for intuitionistic logic were also considered but without success (see discussion in [16, Chapter 4, Section 2]).

In order to see why simple modifications of the method or post-processing of the proofs do not work, take the following example. This is an interesting proof because it is an intuitionistic proof of a formula which is only valid in classical logic. It is only possible because of the assumption of the excluded middle in the antecedent. Let $\varphi$ be the following proof, where cut ancestors are **bold**:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$P \vdash \mathbf{P}$}
\RightLabel{$\vee_r$}
\UnaryInfC{$P \vdash \mathbf{P} \vee \neg\mathbf{P}$}
\AxiomC{}
\UnaryInfC{$\mathbf{P} \vdash P$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\mathbf{P}, \neg P \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg P \vdash \neg\mathbf{P}$}
\RightLabel{$\vee_r$}
\UnaryInfC{$\neg P \vdash \mathbf{P} \vee \neg\mathbf{P}$}
\RightLabel{$\vee_l$}
\BinaryInfC{$P \vee \neg P \vdash \mathbf{P} \vee \neg\mathbf{P}$}
\AxiomC{}
\UnaryInfC{$P \vdash \mathbf{P}$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg\mathbf{P}, P \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg\mathbf{P} \vdash \neg P$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg\mathbf{P}, \neg\neg P \vdash$}
\RightLabel{$w_r$}
\UnaryInfC{$\neg\mathbf{P}, \neg\neg P \vdash P$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$\neg\mathbf{P} \vdash \neg\neg P \rightarrow P$}
\AxiomC{}
\UnaryInfC{$\mathbf{P} \vdash P$}
\RightLabel{$w_l$}
\UnaryInfC{$\mathbf{P}, \neg\neg P \vdash P$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$\mathbf{P} \vdash \neg\neg P \rightarrow P$}
\RightLabel{$\vee_l$}
\BinaryInfC{$\mathbf{P} \vee \neg\mathbf{P} \vdash \neg\neg P \rightarrow P$}
\RightLabel{$cut$}
\BinaryInfC{$P \vee \neg P \vdash \neg\neg P \rightarrow P$}
\end{prooftree}
$$

The characteristic clause set of $\varphi$ is: $\mathrm{CL}(\varphi) = \{\vdash P \;;\; P \vdash \;;\; P \vdash P\}$. Which admits the (only non-redundant) resolution refutation:

$$
\dfrac{\vdash P \quad P \vdash}{\vdash}\; R
$$

The minimal projections are the following:

$$\varphi[\vdash P]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash \mathbf{P}}}{\vdash \mathbf{P}, \neg P}\,\neg_r}{\neg\neg P \vdash \mathbf{P}}\,\neg_l}{\neg\neg P \vdash \mathbf{P}, P}\,w_r}{\vdash \mathbf{P}, \neg\neg P \to P}\,\to_r$$

$$\varphi[P \vdash]$$

$$\dfrac{\dfrac{\dfrac{}{\mathbf{P} \vdash P}}{\mathbf{P}, \neg\neg P \vdash P}\,w_l}{\mathbf{P} \vdash \neg\neg P \to P}\,\to_r$$

Note that $\varphi[\vdash P]$ is classical. The final ACNF is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash \mathbf{P}}}{\vdash \mathbf{P}, \neg P}\,\neg_r}{\neg\neg P \vdash \mathbf{P}}\,\neg_l}{\neg\neg P \vdash \mathbf{P}, P}\,w_r}{\vdash \mathbf{P}, \neg\neg P \to P}\,\to_r \quad \dfrac{\dfrac{\dfrac{}{\mathbf{P} \vdash P}}{\mathbf{P}, \neg\neg P \vdash P}\,w_l}{\mathbf{P} \vdash \neg\neg P \to P}\,\to_r}{\vdash \neg\neg P \to P, \neg\neg P \to P}\,cut}{\vdash \neg\neg P \to P}\,c_r}{P \vee \neg P \vdash \neg\neg P \to P}\,w_l$$

Note that the law of excluded middle is not used at all in the final proof, which means that this proof is inherently classical. Also, the resolution refutation used for `CERES` was the only one possible, which indicates that using resolution refinements would not work. The only thing that could be done was to use the tautological clause $P \vdash P$ in a redundant resolution. Indeed, the method presented in this paper and its completeness proof indicates that, in the general case, this will be necessary.

## 4. Reductive cut-elimination

Now, we introduce a variation of the rewrite rules introduced by Gentzen [10] for eliminating cuts from **LJ**-derivations. In this section, we define a binary relation on **LJ**-derivations based on the reductive cut-elimination rewrite rules. We shall see in Section 5 that, when two **LJ**-derivations are related in this way, the projections constructed from these derivations are also somehow related. This relationship between reductive cut-elimination and projections culminates in one of the main results of this work, Lemma 5.3, which is referred to as the *main subsumption lemma*.

**Definition 4.1 (Cut-elimination rewrite rules).** We define the set $\mathcal{R}$ of rewrite rules for **LJ** as the ones below. In all sequents, $\Delta$ is a set with at most one formula and sets that are possibly modified by the application of an inference rule are annotated with $'$ or $''$.

**Cut-elimination rules:**

Over axiom inferences:

$$\dfrac{\dfrac{}{A \vdash \mathbf{A}} \quad \overset{(\varphi)}{\mathbf{A}, \Gamma \vdash \Delta}}{\Gamma, A \vdash \Delta}\; cut \quad \rightsquigarrow \quad \overset{(\varphi)}{A, \Gamma \vdash \Delta}$$

$$\dfrac{\overset{(\varphi)}{\Gamma \vdash \mathbf{A}} \quad \dfrac{}{\mathbf{A} \vdash A}}{\Gamma \vdash A}\; cut \quad \rightsquigarrow \quad \overset{(\varphi)}{\Gamma \vdash A}$$

Over weakening:

$$\dfrac{\dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash}}{\Gamma_1 \vdash \mathbf{A}}\; w_r \quad \overset{(\varphi_2)}{\mathbf{A}, \Gamma_2 \vdash \Delta}}{\Gamma_1, \Gamma_2 \vdash \Delta}\; cut \quad \rightsquigarrow \quad \genfrac{}{}{1.2pt}{0}{\overset{(\varphi_1)}{\Gamma_1 \vdash}}{\Gamma_1, \Gamma_2 \vdash \Delta}\; w_r^*, w_l^*$$

$$\dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash \mathbf{A}} \quad \dfrac{\overset{(\varphi_2)}{\Gamma_2 \vdash \Delta}}{\mathbf{A}, \Gamma_2 \vdash \Delta}\; w_l}{\Gamma_1, \Gamma_2 \vdash \Delta}\; cut \quad \rightsquigarrow \quad \genfrac{}{}{1.2pt}{0}{\overset{(\varphi_2)}{\Gamma_2 \vdash \Delta}}{\Gamma_1, \Gamma_2 \vdash \Delta}\; w_l^*$$

**Cut-shifting rules:**

Over unary inferences:

$$\dfrac{\dfrac{\overset{(\varphi_1)}{\Gamma_1' \vdash \mathbf{A}}}{\Gamma_1 \vdash \mathbf{A}}\; \rho \quad \overset{(\varphi_2)}{\mathbf{A}, \Gamma_2 \vdash \Delta}}{\Gamma_1, \Gamma_2 \vdash \Delta}\; cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{(\varphi_1)}{\Gamma_1' \vdash \mathbf{A}} \quad \overset{(\varphi_2)}{\mathbf{A}, \Gamma_2 \vdash \Delta}}{\Gamma_1', \Gamma_2 \vdash \Delta}\; cut}{\Gamma_1, \Gamma_2 \vdash \Delta}\; \rho$$

$$\dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash \mathbf{A}} \quad \dfrac{\overset{(\varphi_2)}{\mathbf{A}, \Gamma_2' \vdash \Delta'}}{\mathbf{A}, \Gamma_2 \vdash \Delta}\; \rho}{\Gamma_1, \Gamma_2 \vdash \Delta}\; cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash \mathbf{A}} \quad \overset{(\varphi_2)}{\mathbf{A}, \Gamma_2' \vdash \Delta'}}{\Gamma_1, \Gamma_2' \vdash \Delta'}\; cut}{\Gamma_1, \Gamma_2 \vdash \Delta}\; \rho$$

Over binary inferences:

$$\dfrac{\dfrac{\overset{(\varphi_1)}{P, \Gamma_1 \vdash \mathbf{A}} \quad \overset{(\varphi_2)}{Q, \Gamma_2 \vdash \mathbf{A}}}{P \lor Q, \Gamma_1, \Gamma_2 \vdash \mathbf{A}}\; \lor_l \quad \overset{(\varphi_3)}{\mathbf{A}, \Gamma_3 \vdash \Delta}}{P \lor Q, \Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\; cut \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\overset{(\varphi_1)}{P, \Gamma_1 \vdash \mathbf{A}} \quad \overset{(\varphi_3)}{\mathbf{A}, \Gamma_3 \vdash \Delta}}{P, \Gamma_1, \Gamma_3 \vdash \Delta}\; cut \quad \dfrac{\overset{(\varphi_2)}{Q, \Gamma_2 \vdash \mathbf{A}} \quad \overset{(\varphi_3 \sigma)}{\mathbf{A}, \Gamma_3 \vdash \Delta}}{Q, \Gamma_2, \Gamma_3 \vdash \Delta}\; cut}{P \lor Q, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_3 \vdash \Delta}\; \lor_l}{P \lor Q, \Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\; c_l^*$$

where $\sigma$ is an eigenvariable-renaming making the proof regular[3].

[3] A proof is said to be regular if all eigenvariables used are unique in the whole proof. In general, such a strong requirement is not necessary, but it will be important for the transformations defined later.

The only other possible left binary rule is $\to_l$:

$$\dfrac{\dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash P} \quad \overset{(\varphi_2)}{Q, \Gamma_2 \vdash \mathbf{A}}}{P \to Q, \Gamma_1, \Gamma_2 \vdash \mathbf{A}} \to_l \quad \overset{(\varphi_3)}{\mathbf{A}, \Gamma_3 \vdash \Delta}}{P \to Q, \Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\, \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash P} \quad \dfrac{\overset{(\varphi_2)}{Q, \Gamma_2 \vdash \mathbf{A}} \quad \overset{(\varphi_3)}{\mathbf{A}, \Gamma_3 \vdash \Delta}}{Q, \Gamma_2, \Gamma_3 \vdash \Delta}\, \mathit{cut}}{P \to Q, \Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta} \to_l$$

For a binary rule applied to the right branch of cut, there are no special cases:

$$\dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash \mathbf{A}} \quad \dfrac{\overset{(\varphi_2)}{\mathbf{A}, \Gamma_2' \vdash \Delta'} \quad \overset{(\varphi_3)}{\Gamma_3' \vdash \Delta''}}{\mathbf{A}, \Gamma_2, \Gamma_3 \vdash \Delta}\, \rho}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\, \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash \mathbf{A}} \quad \overset{(\varphi_2)}{\mathbf{A}, \Gamma_2' \vdash \Delta'}}{\Gamma_1, \Gamma_2' \vdash \Delta'}\, \mathit{cut} \quad \overset{(\varphi_3)}{\Gamma_3' \vdash \Delta''}}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\, \rho$$

$$\dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash \mathbf{A}} \quad \dfrac{\overset{(\varphi_2)}{\Gamma_2' \vdash \Delta'} \quad \overset{(\varphi_3)}{\mathbf{A}, \Gamma_3' \vdash \Delta''}}{\mathbf{A}, \Gamma_2, \Gamma_3 \vdash \Delta}\, \rho}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\, \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{(\varphi_2)}{\Gamma_2' \vdash \Delta'} \quad \dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash \mathbf{A}} \quad \overset{(\varphi_3)}{\mathbf{A}, \Gamma_3' \vdash \Delta''}}{\Gamma_1, \Gamma_3' \vdash \Delta''}\, \mathit{cut}}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\, \rho$$

**Cut-simplification rules:**

Of a cut on an $\wedge$-formula:

$$\dfrac{\dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash \mathbf{A}} \quad \overset{(\varphi_2)}{\Gamma_2 \vdash \mathbf{B}}}{\Gamma_1, \Gamma_2 \vdash \mathbf{A} \wedge \mathbf{B}}\, \wedge_r \quad \dfrac{\overset{(\varphi_3)}{\mathbf{A}, \Gamma_3 \vdash \Delta}}{\mathbf{A} \wedge \mathbf{B}, \Gamma_3 \vdash \Delta}\, \wedge_{l1}}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\, \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash \mathbf{A}} \quad \overset{(\varphi_3)}{\mathbf{A}, \Gamma_3 \vdash \Delta}}{\Gamma_1, \Gamma_3 \vdash \Delta}\, \mathit{cut}}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\, w_l^*$$

$$\dfrac{\dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash \mathbf{A}} \quad \overset{(\varphi_2)}{\Gamma_2 \vdash \mathbf{B}}}{\Gamma_1, \Gamma_2 \vdash \mathbf{A} \wedge \mathbf{B}}\, \wedge_r \quad \dfrac{\overset{(\varphi_3)}{\mathbf{B}, \Gamma_3 \vdash \Delta}}{\mathbf{A} \wedge \mathbf{B}, \Gamma_3 \vdash \Delta}\, \wedge_{l2}}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\, \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{(\varphi_2)}{\Gamma_2 \vdash \mathbf{B}} \quad \overset{(\varphi_3)}{\mathbf{B}, \Gamma_3 \vdash \Delta}}{\Gamma_2, \Gamma_3 \vdash \Delta}\, \mathit{cut}}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\, w_l^*$$

Of a cut on an $\vee$-formula:

$$\dfrac{\dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash \mathbf{A}}}{\Gamma_1 \vdash \mathbf{A} \vee \mathbf{B}}\, \vee_{r1} \quad \dfrac{\overset{(\varphi_2)}{\mathbf{A}, \Gamma_2 \vdash \Delta} \quad \overset{(\varphi_3)}{\mathbf{B}, \Gamma_3 \vdash \Delta}}{\mathbf{A} \vee \mathbf{B}, \Gamma_2, \Gamma_3 \vdash \Delta}\, \vee_l}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\, \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash \mathbf{A}} \quad \overset{(\varphi_2)}{\mathbf{A}, \Gamma_2 \vdash \Delta}}{\Gamma_1, \Gamma_2 \vdash \Delta}\, \mathit{cut}}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\, w_l^*$$

$$\dfrac{\dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash \mathbf{B}}}{\Gamma_1 \vdash \mathbf{A} \vee \mathbf{B}}\, \vee_{r2} \quad \dfrac{\overset{(\varphi_2)}{\mathbf{A}, \Gamma_2 \vdash \Delta} \quad \overset{(\varphi_3)}{\mathbf{B}, \Gamma_3 \vdash \Delta}}{\mathbf{A} \vee \mathbf{B}, \Gamma_2, \Gamma_3 \vdash \Delta}\, \vee_l}{\Gamma, \Gamma_2, \Gamma_3 \vdash \Delta}\, \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash \mathbf{B}} \quad \overset{(\varphi_3)}{\mathbf{B}, \Gamma_3 \vdash \Delta}}{\Gamma_1, \Gamma_3 \vdash \Delta}\, \mathit{cut}}{\Gamma, \Gamma_2, \Gamma_3 \vdash \Delta}\, w_l^*$$

Of a cut on an $\to$-formula:

$$\dfrac{\dfrac{\overset{(\varphi_1)}{\Gamma_1, \mathbf{A} \vdash \mathbf{B}}}{\Gamma_1 \vdash \mathbf{A} \to \mathbf{B}}\ \to_r \qquad \dfrac{\overset{(\varphi_2)}{\Gamma_2 \vdash \mathbf{A}} \quad \overset{(\varphi_3)}{\mathbf{B}, \Gamma_3 \vdash \Delta}}{\mathbf{A} \to \mathbf{B}, \Gamma_2, \Gamma_3 \vdash \Delta}\ \to_l}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\ \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{(\varphi_2)}{\Gamma_2 \vdash \mathbf{A}} \qquad \dfrac{\overset{(\varphi_1)}{\mathbf{A}, \Gamma_1 \vdash \mathbf{B}} \quad \overset{(\varphi_3)}{\mathbf{B}, \Gamma_3 \vdash \Delta}}{\mathbf{A}, \Gamma_1, \Gamma_3 \vdash \Delta}\ \mathit{cut}}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\ \mathit{cut}$$

Of a cut on an $\neg$-formula:

$$\dfrac{\dfrac{\overset{(\varphi_1)}{\Gamma_1, \mathbf{A} \vdash}}{\Gamma_1 \vdash \neg\mathbf{A}}\ \neg_r \qquad \dfrac{\overset{(\varphi_2)}{\Gamma_2 \vdash \mathbf{A}}}{\neg\mathbf{A}, \Gamma_2 \vdash}\ \neg_l}{\Gamma_1, \Gamma_2 \vdash}\ \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{(\varphi_2)}{\Gamma_2 \vdash \mathbf{A}} \quad \overset{(\varphi_1)}{\mathbf{A}, \Gamma_1 \vdash}}{\Gamma_1, \Gamma_2 \vdash}\ \mathit{cut}$$

Of a cut on an $\exists$-formula:

$$\dfrac{\dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash \mathbf{A}[\mathbf{x}/\mathbf{t}]}}{\Gamma_1 \vdash \exists \mathbf{x}.\mathbf{Ax}}\ \exists_r \qquad \dfrac{\overset{(\varphi_2)}{\mathbf{A}[\mathbf{x}/\alpha], \Gamma_2 \vdash \Delta}}{\exists \mathbf{x}.\mathbf{Ax}, \Gamma_2 \vdash \Delta}\ \exists_l}{\Gamma_1, \Gamma_2 \vdash \Delta}\ \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash \mathbf{A}[\mathbf{x}/\mathbf{t}]} \quad \overset{(\varphi_2[\alpha/t])}{\mathbf{A}[\mathbf{x}/\mathbf{t}], \Gamma_2 \vdash \Delta}}{\Gamma_1, \Gamma_2 \vdash \Delta}\ \mathit{cut}$$

Of a cut on an $\forall$-formula:

$$\dfrac{\dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash \mathbf{A}[\mathbf{x}/\alpha]}}{\Gamma_1 \vdash \forall \mathbf{x}.\mathbf{Ax}}\ \forall_r \qquad \dfrac{\overset{(\varphi_2)}{\mathbf{A}[\mathbf{x}/\mathbf{t}], \Gamma_2 \vdash \Delta}}{\forall \mathbf{x}.\mathbf{Ax}, \Gamma_2 \vdash \Delta}\ \forall_l}{\Gamma_1, \Gamma_2 \vdash \Delta}\ \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{(\varphi_1[\alpha/t])}{\Gamma_1 \vdash \mathbf{A}[\mathbf{x}/\mathbf{t}]} \quad \overset{(\varphi_2)}{\mathbf{A}[\mathbf{x}/\mathbf{t}], \Gamma_2 \vdash \Delta}}{\Gamma_1, \Gamma_2 \vdash \Delta}\ \mathit{cut}$$

Of a cut on a contracted formula:

$$\dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash \mathbf{A}} \qquad \dfrac{\overset{(\varphi_2)}{\mathbf{A}, \mathbf{A}, \Gamma_2 \vdash \Delta}}{\mathbf{A}, \Gamma_2 \vdash \Delta}\ c_l}{\Gamma_1, \Gamma_2 \vdash \Delta}\ \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{(\varphi_1)}{\Gamma_1 \vdash \mathbf{A}} \qquad \dfrac{\overset{(\varphi_1\sigma)}{\Gamma_1 \vdash \mathbf{A}} \quad \overset{(\varphi_2)}{\mathbf{A}, \mathbf{A}, \Gamma_2 \vdash \Delta}}{\mathbf{A}, \Gamma_1, \Gamma_2 \vdash \Delta}\ \mathit{cut}}{\Gamma_1, \Gamma_1, \Gamma_2 \vdash \Delta}\ \mathit{cut}}{\Gamma_1, \Gamma_2 \vdash \Delta}\ c_l^*$$

where $\sigma$ is an eigenvariable-renaming making the proof regular.

**Definition 4.2.** Let $\varphi, \psi$ be **LJ**-proofs. We define $\varphi \rightsquigarrow_{\mathcal{R}'} \psi$:

- if $\varphi$ rewrites in one step to $\psi$ according to the cut-elimination rewrite rules specified in $\mathcal{R}' \subseteq \mathcal{R}$; or
- if there exists a node $\nu$ in $\varphi$ such that $\varphi.\nu \rightsquigarrow_{\mathcal{R}'} \chi$ and $\psi = \varphi[\chi]_\nu$.

We call $\rightsquigarrow_{\mathcal{R}'}$ a *cut-reduction relation*. The reflexive transitive closure of $\rightsquigarrow_{\mathcal{R}'}$ is denoted by $\rightsquigarrow^*_{\mathcal{R}'}$.

For this work, we will be interested in a particular subset of $\mathcal{R}$:

**Definition 4.3.** We will denote by $\mathcal{R}^a$ the set of rules $\mathcal{R}$ without the cut-elimination rules over atomic axioms.

The notation used for this set is motivated by the fact that, using these rewrite rules, normalization always results in a proof with atomic cuts as the top-most inferences.

**Definition 4.4 ($\mathrm{ACNF}, \mathrm{ACNF}^{\mathrm{top}}$).** Let $\varphi$ be an **LJ**-derivation. We say that $\varphi$ is in *atomic cut normal form* (ACNF) if it contains only atomic cuts. We say that $\varphi$ is in *top atomic cut normal form* ($\mathrm{ACNF}^{\mathrm{top}}$) if it is in ACNF and it is irreducible under $\rightsquigarrow_{\mathcal{R}^a}$.

Top atomic cut normal forms will play a crucial role in the completeness proof of intuitionistic `CERES`.

## 5. Proof Subsumption

We first extend the subsumption concept from clauses to full sequents. Basically, the definition is the same, with the exception that we consider $\subseteq$ as *multiset* inclusion, instead of set inclusion. In a second step, we define a notion of subsumption for **LK**-proofs.

**Definition 5.1 (Sequent subsumption).** Let $S\colon \Gamma \vdash \Delta$ and $S'\colon \Gamma' \vdash \Delta'$. We define $S \subseteq S'$ if $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$ ($\subseteq$ denotes the multiset inclusion). Moreover, we say that $S$ subsumes $S'$ via $\vartheta$ if there exists a substitution $\vartheta$ on first-order variables such that $S\vartheta \subseteq S'$.

Proof subsumption is basically a generalization of the subsumption concept for resolution proofs [14]. It is, however, much more involved as **LK**-proofs have more rules. It is defined inductively on the number of inferences for proofs without strong quantifier inferences.

**Definition 5.2 (Proof subsumption (sketch)).** Let $\varphi$ and $\psi$ be **LK**-proofs without cuts or strong quantifier inferences and $\vartheta$ a substitution on first-order variables. We say that $\varphi \leq_{ss} \psi$ via $\vartheta$, or equivalently $(\varphi, \psi, \vartheta)$ is a proof subsumption, if:

**Base case:** $\varphi$ is an axiom $S$ and $S\vartheta \subseteq \mathrm{es}(\psi)$. Note that, if $\psi$ is an axiom, then $\varphi$ must be an axiom as well.

**Inductive cases:** Let $(\varphi_i, \psi_i, \vartheta)$ be proof subsumptions with $\mathrm{es}(\varphi_i) = \Gamma_i \vdash \Delta_i$ and $\mathrm{es}(\psi_i) = \Pi_i \vdash \Lambda_i$:

- Assume that an inference $\rho_2$ is applied to a finite subset $\{\psi_1, ..., \psi_k\}$ of proofs, resulting in a proof $\psi$ with end-sequent $\Pi \vdash \Lambda$. Then $(\varphi_i, \psi, \vartheta)$, for some $1 \leq i \leq k$, is a proof subsumption iff $(\Gamma_i \vdash \Delta_i)\vartheta \subseteq \Pi \vdash \Lambda$,
- Assume that an inference $\rho_1$ is applied to a finite subset $\{\varphi_1, ..., \varphi_k\}$ of proofs, resulting in a proof $\varphi$ with end-sequent $\Gamma \vdash \Delta$. Then $(\varphi, \psi_i, \vartheta')$, for some $1 \leq i \leq k$ and substitution $\vartheta'$, is a proof subsumption iff $(\Gamma \vdash \Delta)\vartheta' \subseteq \Pi_i \vdash \Lambda_i$.
- Assume that an inference $\rho_1$ is applied to a finite subset $\{\varphi_1, ..., \varphi_k\}$ of proofs, resulting in a proof $\varphi$ with end-sequent $\Gamma \vdash \Delta$, and an inference $\rho_2$ is applied to a finite subset $\{\psi_1, ..., \psi_k\}$ of proofs, resulting in a proof $\psi$ with end-sequent $\Pi \vdash \Lambda$. Then $(\varphi, \psi, \vartheta')$ for some substitution $\vartheta'$ is a proof subsumption iff $(\Gamma \vdash \Delta)\vartheta' \subseteq \Pi \vdash \Lambda$.

**Definition 5.3 (Proof subsumption (detailed)).** Below we define the relation $\varphi \leq_{ss} \psi$ ($\varphi$ subsumes $\psi$) for cut-free **LK**-proofs $\varphi$ and $\psi$ without strong quantifier inferences.

**Base case:** Let $\varphi$ be an axiom $S$. Then $\varphi \leq_{ss} \psi$ via $\vartheta$ (or $(\varphi, \psi, \vartheta)$ is a proof subsumption) iff $S\vartheta \subseteq \mathrm{es}(\psi)$.

**Inductive cases:** Assume that $(\varphi_i, \psi_i, \vartheta_i)$, $i \in \{1, 2\}$, are proof subsumptions. We distinguish cases based on the rule $\rho_2$ applied to $\psi_i$ and, for each possibility, analyse the cases for $\varphi_i$.

1. $\neg_r$: Then $\psi$ is

$$\dfrac{\begin{array}{c}(\psi_1)\\ A, \Pi \vdash \Lambda\end{array}}{\Pi \vdash \Lambda, \neg A}\ \neg_r$$

Let $\mathrm{es}(\varphi_1) = \Gamma \vdash \Delta$. We distinguish two cases:

(a) $(\Gamma \vdash \Delta)\vartheta_1 \subseteq \Pi \vdash \Lambda$
Then $(\varphi_1, \psi, \vartheta_1)$ is a proof subsumption.
Additionally, we can define $\varphi$ as:

$$\dfrac{\begin{array}{c}(\varphi_1)\\ \Gamma \vdash \Delta\end{array}}{\Gamma \vdash \Delta, \neg A_0}\ w_r$$

where $A_0\vartheta_1 = A$ and then also $(\varphi, \psi, \vartheta_1)$ is a proof subsumption.

(b) $(\Gamma \vdash \Delta)\vartheta_1 \not\subseteq \Pi \vdash \Lambda$
By the induction hypothesis we know that $(\Gamma \vdash \Delta)\vartheta_1 \subseteq A, \Pi \vdash \Lambda$. Then it must be the case that $\Gamma = A_0, \Gamma'$ with $(\Gamma' \vdash \Delta)\vartheta_1 \subseteq \Pi \vdash \Lambda$ and $A_0\vartheta_1 = A$.
In this case we define $\varphi$:

$$\dfrac{\begin{array}{c}(\varphi_1)\\ A_0, \Gamma \vdash \Delta\end{array}}{\Gamma \vdash \Delta, \neg A_0}\ \neg_r$$

and $(\varphi, \psi, \vartheta_1)$ is a proof subsumption.

2. $\neg_l$: analogous to $\neg_r$.

3. $\rightarrow_r$: Then $\psi$ is

$$\dfrac{\begin{array}{c}(\psi_1)\\ A,\Pi\vdash\Lambda,B\end{array}}{\Pi\vdash\Lambda,A\rightarrow B}\ \rightarrow_r$$

We assume that $V(\varphi_1)\cap V(\psi_1)=\emptyset$[4] and $dom(\vartheta_1)\subseteq V(\varphi_1)$. Let $\mathrm{es}(\varphi_1)=\Gamma\vdash\Delta$. We distinguish several cases:

(a) $(\Gamma\vdash\Delta)\vartheta_1\subseteq\Pi\vdash\Lambda$
Then $(\varphi_1,\psi,\vartheta_1)$ is a proof subsumption.
Additionally, we can define $\varphi$ as

$$\dfrac{\begin{array}{c}(\varphi_1)\\ \Gamma\vdash\Delta\end{array}}{\Gamma\vdash\Delta,A_0\rightarrow B_0}\ w_r$$

where $(A_0\rightarrow B_0)\vartheta_1=(A\rightarrow B)$ and then also $(\varphi,\psi,\vartheta_1)$ is a proof subsumption.

(b) $(\Gamma\vdash\Delta)\vartheta_1\not\subseteq\Pi\vdash\Lambda$

i. $(\Gamma\vdash\Delta)\vartheta_1\subseteq A,\Pi\vdash\Lambda$
Then $\Gamma=A_0,\Gamma'$ such that $(\Gamma'\vdash\Delta)\vartheta_1\subseteq\Pi\vdash\Lambda$ and $A_0\vartheta=A$. We thus define $\varphi$ as

$$\dfrac{\dfrac{\begin{array}{c}(\varphi_1)\\ A_0,\Gamma'\vdash\Delta\end{array}}{A_0,\Gamma'\vdash\Delta,B}\ w_r}{\Gamma'\vdash\Delta,A_0\rightarrow B}\ \rightarrow_r$$

and $(\varphi,\psi,\vartheta_1)$ is a proof subsumption. Note that $B\vartheta_1=B$ as $B$ occurs in $\psi_1$ and $V(\psi_1)\cap dom(\vartheta_1)=\emptyset$. So $(\Gamma'\vdash\Delta,A_0\rightarrow B)\vartheta_1\subseteq\Pi\vdash\Lambda,A\rightarrow B$.

ii. $(\Gamma\vdash\Delta)\vartheta_1\subseteq\Pi\vdash\Lambda,B$ Then $\Delta=\Delta',B_0$ such that $(\Gamma\vdash\Delta')\vartheta_1\subseteq\Pi\vdash\Lambda$ and $B_0\vartheta_1=B$. We thus define $\varphi$ as

$$\dfrac{\dfrac{\begin{array}{c}(\varphi_1)\\ \Gamma\vdash\Delta',B_0\end{array}}{A,\Gamma\vdash\Delta',B_0}\ w_l}{\Gamma\vdash\Delta',A\rightarrow B_0}\ \rightarrow_r$$

and $(\varphi,\psi,\vartheta)$ is a proof subsumption.

iii. Neither 3(b)i nor 3(b)ii hold.

[4] $V(\cdot)$ denotes free variables. See preliminaries for further details, Section 2

Then $\Gamma = A_0, \Gamma'$ and $\Delta = \Delta', B_0$ with $(\Gamma' \vdash \Delta')\vartheta_1 \subseteq \Pi \vdash \Lambda$ and $A_0\vartheta_1 = A$, $B_0\vartheta_1 = B$. We define $\varphi$ as

$$\cfrac{\begin{matrix}(\varphi_1)\\ A_0, \Gamma' \vdash \Delta', B_0\end{matrix}}{\Gamma' \vdash \Delta', A_0 \to B_0}\ \to_r$$

and $(\varphi, \psi, \vartheta)$ is a proof subsumption.

4. $\wedge_r$: Then $\psi$ is

$$\cfrac{\begin{matrix}(\psi_1)\\ \Pi_1 \vdash \Lambda_1, A\end{matrix} \quad \begin{matrix}(\psi_2)\\ \Pi_2 \vdash \Lambda_2, B\end{matrix}}{\Pi_1, \Pi_2 \vdash \Lambda_1, \Lambda_2, A \wedge B}\ \wedge_r$$

We assume that $V(\varphi_i) \cap V(\psi_i) = \emptyset$ and $dom(\vartheta_i) = V(\varphi_i)$ for $i \in \{1, 2\}$ and thus $\vartheta = \vartheta_1 \cup \vartheta_2$ is well-defined.
Let $\mathrm{es}(\varphi_1) = \Gamma_1 \vdash \Delta_1$ and $\mathrm{es}(\varphi_2) = \Gamma_2 \vdash \Delta_2$. We distinguish 3 cases:

(a) $(\Gamma_1 \vdash \Delta_1)\vartheta_1 \subseteq \Pi_1 \vdash \Lambda_1$.
Then $(\varphi_1, \psi, \vartheta_1)$ is a proof subsumption.
Additionally, let $\Gamma^*\vartheta \subseteq \Pi_2$ and $\Delta^*\vartheta \subseteq \Lambda_2$. We can define $\varphi$ as

$$\cfrac{\begin{matrix}(\varphi_1)\\ \Gamma_1 \vdash \Delta_1\end{matrix}}{\Gamma_1, \Gamma^* \vdash \Delta_1, \Delta^*}\ w_l^*, w_r^*$$

Then also $(\varphi, \psi, \vartheta)$ is a proof subsumption.

(b) $(\Gamma_2 \vdash \Delta_2)\vartheta_2 \subseteq \Pi_2 \vdash \Lambda_2$.
Then $(\varphi_2, \psi, \vartheta_2)$ is a proof subsumption.
Additionally, let $\Gamma^*\vartheta \subseteq \Pi_1$ and $\Delta^*\vartheta \subseteq \Lambda_1$. We can define $\varphi$ as

$$\cfrac{\begin{matrix}(\varphi_2)\\ \Gamma_2 \vdash \Delta_2\end{matrix}}{\Gamma^*, \Gamma_2 \vdash \Delta^*, \Delta_2}\ w_l^*, w_r^*$$

Then also $(\varphi, \psi, \vartheta)$ is a proof subsumption.

(c) Neither 4a nor 4b hold.
Then $\Delta_1 = \Delta_1', A_0$ and $\Delta_2 = \Delta_2', B_0$ such that $(\Gamma_1 \vdash \Delta_1')\vartheta_1 \subseteq \Pi_1 \vdash \Lambda_1$ and $(\Gamma_2 \vdash \Delta_2')\vartheta_1 \subseteq \Pi_2 \vdash \Lambda_2$, and $A_0\vartheta_1 = A, B_0\vartheta_2 = B$. We define $\varphi$ as

$$\cfrac{\begin{matrix}(\varphi_1)\\ \Gamma_1 \vdash \Delta_1', A_0\end{matrix} \quad \begin{matrix}(\varphi_2)\\ \Gamma_2 \vdash \Delta_2', B_0\end{matrix}}{\Gamma_1, \Gamma_2 \vdash \Delta_1', \Delta_2', A_0 \wedge B_0}\ \wedge_r$$

and $(\varphi, \psi, \vartheta)$ as proof subsumption.

5. $\to_l$ and $\vee_l$: analogous to $\wedge_r$.

6. $\forall_l$: Then $\psi$ is

$$\cfrac{\begin{matrix}(\psi_1)\\ A\{x \leftarrow t\}, \Pi \vdash \Lambda\end{matrix}}{\forall x.A, \Pi \vdash \Lambda}\ \forall_l$$

Let $\mathrm{es}(\varphi_1) = \Gamma \vdash \Delta$. We distinguish two cases:

(a) $(\Gamma \vdash \Delta)\vartheta_1 \subseteq \Pi \vdash \Lambda$.
Then $(\varphi_1, \psi, \vartheta_1)$ is a proof subsumption.
Additionally, let $A_0$ be a formula such that $(\forall x.A_0)\vartheta_1 = \forall x.A$. We can define $\varphi$ as

$$\dfrac{\begin{array}{c}(\varphi_1)\\ \Gamma \vdash \Delta\end{array}}{\forall x.A_0, \Gamma \vdash \Delta}\; w_l$$

and $(\varphi, \psi, \vartheta_1)$ is a proof subsumption.

(b) $(\Gamma \vdash \Delta)\vartheta_1 \not\subseteq \Pi \vdash \Lambda$.
Then $\Gamma = A_0\{x \leftarrow s\}, \Gamma'$ such that $(\Gamma' \vdash \Delta)\vartheta_1 \subseteq \Pi \vdash \Lambda$ and $A_0\{x \leftarrow s\}\vartheta_1 = A\{x \leftarrow t\}$. We define $\varphi$ as

$$\dfrac{\begin{array}{c}(\varphi_1)\\ A_0\{x \leftarrow s\}, \Gamma' \vdash \Delta\end{array}}{\forall x.A_0, \Gamma' \vdash \Delta}\; \forall_l$$

and $(\varphi, \psi, \vartheta_1)$ is a proof subsumption.

7. $\exists_r$: analogous to $\forall_l$.

8. $w_l$: Then $\psi$ is

$$\dfrac{\begin{array}{c}(\psi_1)\\ \Pi \vdash \Lambda\end{array}}{A, \Pi \vdash \Lambda}\; w_l$$

In this case, $(\varphi_1, \psi, \vartheta_1)$ is a proof subsumption.
Additionally, let $\mathrm{es}(\varphi_1) = \Gamma \vdash \Delta$ and $A_0$ a formula with $A_0\vartheta_1 = A$. We can define $\varphi$ as

$$\dfrac{\begin{array}{c}(\varphi_1)\\ \Gamma \vdash \Delta\end{array}}{A_0, \Gamma \vdash \Delta}\; w_l$$

Then also $(\varphi, \psi, \vartheta_1)$ is a proof subsumption.

9. $w_r$: analogous to $w_l$.

10. $c_l$: Then $\psi$ is

$$\dfrac{\begin{array}{c}(\psi_1)\\ A, A, \Pi \vdash \Lambda\end{array}}{A, \Pi \vdash \Lambda}\; c_l$$

Let $\mathrm{es}(\varphi_1) = \Gamma \vdash \Delta$. We distinguish two cases:

(a) $(\Gamma \vdash \Delta)\vartheta_1 \subseteq A, \Pi \vdash \Lambda$.
Then $(\varphi_1, \psi, \vartheta_1)$ is a proof subsumption.

(b) $(\Gamma \vdash \Delta)\vartheta_1 \not\subseteq A, \Pi \vdash \Lambda$.
Then $\Gamma = A_0, A_1, \Gamma'$ such that $(\Gamma' \vdash \Delta)\vartheta_1 \subseteq \Pi \vdash \Lambda$ and $A_0\vartheta_1 = A$ and $A_1\vartheta_1 = A$. Therefore, $\vartheta_1$ is a unifier of $\{A_0, A_1\}$, and so there exists a most general unifier $\sigma$ of $\{A_0, A_1\}$. In particular, there exists a substitution $\eta$ such that $\vartheta_1 = \sigma\eta$. We define $\varphi$ as

$$\dfrac{\begin{array}{c}(\varphi_1\sigma)\\ A_0\sigma, A_1\sigma, \Gamma'\sigma \vdash \Delta\sigma\end{array}}{A\sigma, \Gamma'\sigma \vdash \Delta\sigma}\; c_l$$

and $(\varphi, \psi, \eta)$ is a proof subsumption.

11. $c_r$: analogous to $c_l$.

This completes the definition of proof subsumption.

We need proof subsumptions only for projections and their grounded resolutions (to be defined in Section 6). Therefore, there is no need to take care of strong quantifier inferences or cuts.

**Example 5.1.** Let $\varphi =$

$$\cfrac{\cfrac{\cfrac{\cfrac{P(x) \vdash P(x)}{\neg Q(x), P(x) \vdash P(x)}\ w_l}{\neg Q(x) \vdash P(x), \neg P(x)}\ \neg_r}{\vdash P(x), \neg Q(x) \to \neg P(x)}\ \to_r}{\vdash P(x), \exists z.(\neg Q(z) \to \neg P(z))}\ \exists_r$$

and $\psi =$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{P(f(a)) \vdash P(f(a))}{\neg P(f(a)), P(f(a)) \vdash P(f(a))}\ w_l \quad \cfrac{Q(f(a)) \vdash Q(f(a))}{\neg Q(f(a)), Q(f(a)) \vdash Q(f(a))}\ w_l}{P(f(a)), \neg P(f(a)) \vee Q(f(a)), \neg Q(f(a)) \vdash P(f(a)), Q(f(a))}\ \vee_l}{\neg P(f(a)) \vee Q(f(a)), \neg Q(f(a)) \vdash \neg P(f(a)), P(f(a)), Q(f(a))}\ \neg_r}{\forall x(\neg P(x) \vee Q(x)), \neg Q(f(a)) \vdash \neg P(f(a)), P(f(a)), Q(f(a))}\ \forall_l}{\forall x(\neg P(x) \vee Q(x)) \vdash \neg Q(f(a)) \to \neg P(f(a)), P(f(a)), Q(f(a))}\ \to_r}{\forall x(\neg P(x) \vee Q(x)) \vdash \exists z(\neg Q(z) \to \neg P(z)), P(f(a)), Q(f(a))}\ \exists_r$$

We show that $(\varphi, \psi, \vartheta)$ for $\vartheta = \{x \leftarrow f(a)\}$ is a proof subsumption.

By definition the proof $\varphi_1 \colon P(x) \vdash P(x)$ subsumes $\psi_1 \colon P(f(a)) \vdash P(f(a))$ and $(\varphi_1, \psi_1, \vartheta)$ is a proof subsumption. Let $\psi_2 =$

$$\cfrac{P(f(a)) \vdash P(f(a))}{\neg P(f(a)), P(f(a)) \vdash P(f(a))}\ w_l$$

Then, by the case 8 for $w_l$, $(\varphi_1, \psi_2, \vartheta)$ is a proof subsumption. Now consider $\psi_3 =$

$$\cfrac{\cfrac{P(f(a)) \vdash P(f(a))}{\neg P(f(a)), P(f(a)) \vdash P(f(a))}\ w_l \quad \cfrac{Q(f(a)) \vdash Q(f(a))}{\neg Q(f(a)), Q(f(a)) \vdash Q(f(a))}\ w_l}{P(f(a)), \neg P(f(a)) \vee Q(f(a)), \neg Q(f(a)) \vdash P(f(a)), Q(f(a))}\ \vee_l$$

We define $\varphi_2 =$

$$\cfrac{P(x) \vdash P(x)}{\neg Q(x), P(x) \vdash P(x)}\ w_l$$

By the $\vee_l$-case 5 in the subsumption definition, we get that $(\varphi_2, \psi_3, \vartheta)$ is a proof subsumption.

Now consider $\psi_4 =$

$$\dfrac{\dfrac{\dfrac{P(f(a)) \vdash P(f(a))}{\neg P(f(a)), P(f(a)) \vdash P(f(a))}\, w_l \qquad \dfrac{Q(f(a)) \vdash Q(f(a))}{\neg Q(f(a)), Q(f(a)) \vdash Q(f(a))}\, w_l}{P(f(a)), \neg P(f(a)) \vee Q(f(a)), \neg Q(f(a)) \vdash P(f(a)), Q(f(a))}\, \vee_l}{\neg P(f(a)) \vee Q(f(a)), \neg Q(f(a)) \vdash \neg P(f(a)), P(f(a)), Q(f(a))}\, \neg_r$$

Let $\varphi_3 =$

$$\dfrac{\dfrac{P(x) \vdash P(x)}{\neg Q(x), P(x) \vdash P(x)}\, w_l}{\neg Q(x) \vdash P(x), \neg P(x)}\, \neg_r$$

By $\neg_r$ case 1b we get $(\varphi_3, \psi_4, \vartheta)$ as a proof subsumption.

Let $\psi_5 =$

$$\dfrac{\dfrac{\dfrac{\dfrac{P(f(a)) \vdash P(f(a))}{\neg P(f(a)), P(f(a)) \vdash P(f(a))}\, w_l \qquad \dfrac{Q(f(a)) \vdash Q(f(a))}{\neg Q(f(a)), Q(f(a)) \vdash Q(f(a))}\, w_l}{P(f(a)), \neg P(f(a)) \vee Q(f(a)), \neg Q(f(a)) \vdash P(f(a)), Q(f(a))}\, \vee_l}{\neg P(f(a)) \vee Q(f(a)), \neg Q(f(a)) \vdash \neg P(f(a)), P(f(a)), Q(f(a))}\, \neg_r}{\forall x(\neg P(x) \vee Q(x)), \neg Q(f(a)) \vdash \neg P(f(a)), P(f(a)), Q(f(a))}\, \forall_l$$

Then by $\forall_l$ case 6a we get $(\varphi_3, \psi_5, \vartheta)$ as a proof subsumption.

Let $\psi_6 =$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(f(a)) \vdash P(f(a))}{\neg P(f(a)), P(f(a)) \vdash P(f(a))}\, w_l \qquad \dfrac{Q(f(a)) \vdash Q(f(a))}{\neg Q(f(a)), Q(f(a)) \vdash Q(f(a))}\, w_l}{P(f(a)), \neg P(f(a)) \vee Q(f(a)), \neg Q(f(a)) \vdash P(f(a)), Q(f(a))}\, \vee_l}{\neg P(f(a)) \vee Q(f(a)), \neg Q(f(a)) \vdash \neg P(f(a)), P(f(a)), Q(f(a))}\, \neg_r}{\forall x(\neg P(x) \vee Q(x)), \neg Q(f(a)) \vdash \neg P(f(a)), P(f(a)), Q(f(a))}\, \forall_l}{\forall x(\neg P(x) \vee Q(x)) \vdash \neg Q(f(a)) \to \neg P(f(a)), P(f(a)), Q(f(a))}\, \to_r$$

We define $\varphi_4 =$

$$\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{\neg Q(x), P(x) \vdash P(x)}\, w_l}{\neg Q(x) \vdash P(x), \neg P(x)}\, \neg_r}{\vdash P(x), \neg Q(x) \to \neg P(x)}\, \to_r$$

By $\to_r$ case 3(b)ii we get $(\varphi_4, \psi_6, \vartheta)$ as a proof subsumption.

Finally, by $\exists_r$ case analogous to 6b, $(\varphi, \psi, \vartheta)$ is a proof subsumption.

**Proposition 5.1** (Transitivity of proof subsumption)**.** *Let* $(\varphi,\ \psi,\ \vartheta)$ *and* $(\psi,\ \chi,\ \lambda)$ *be proof subsumptions such that* $V(\varphi) \cap V(\psi) = \emptyset$ *and* $dom(\vartheta) \subseteq V(\varphi)$, $dom(\lambda) \subseteq V(\psi)$. *Then* $(\varphi, \chi, \vartheta\lambda)$ *is a proof subsumption.*

*Proof.* Let us consider the case when $\varphi$ and $\psi$ are axioms and $\chi$ is an arbitrary **LK**-proof. We know, by definition, $(\psi, \chi, \lambda)$ is equivalent to saying that $\mathrm{es}(\psi)$ (which is just the axiom) subsumes $\mathrm{es}(\chi)$ via $\lambda$, i.e., $\mathrm{es}(\psi)\lambda \subseteq \mathrm{es}(\chi)$. We also know that $\mathrm{es}(\varphi)\vartheta \subseteq \mathrm{es}(\psi)$ (which are both just the axioms). Since sequent subsumption is transitive, we have that $\mathrm{es}(\varphi)\vartheta\lambda \subseteq \mathrm{es}(\chi)$. Therefore, $(\varphi, \chi, \vartheta\lambda)$ is a proof subsumption.

Now let us assume that $\varphi$ is an axiom and both $\psi$ and $\chi$ are arbitrary **LK**-proofs. We know, by definition, that $\mathrm{es}(\varphi)\vartheta \subseteq \mathrm{es}(\psi)$ and $\mathrm{es}(\psi)\lambda \subseteq \mathrm{es}(\chi)$. Again, by transitivity, we have that $\mathrm{es}(\varphi)\vartheta\lambda \subseteq \mathrm{es}(\chi)$, where $\mathrm{es}(\varphi)$ is simply an axiom. By the definition of proof subsumption (base case), we thus have that $(\varphi, \chi, \vartheta\lambda)$ is a proof subsumption.

Now let us consider the case when $\varphi$, $\psi$, and $\chi$ are arbitrary **LK**-proofs. We know by the definition of proof subsumption that the proof subsumption $(\varphi, \psi, \vartheta)$ is constructed from base case proof subsumptions $(A_1^\varphi, \psi_1^{sub}, \vartheta_1), \cdots,$ $(A_n^\varphi, \psi_n^{sub}, \vartheta_n)$, where the $A_i^\varphi$ are axioms. Also, $(\psi, \chi, \lambda)$ can be deconstructed into proof subsumptions $(\psi_1^{sub}, \chi_1^{sub}, \lambda_1), \cdots, (\psi_n^{sub}, \chi_n^{sub}, \lambda_n)$. By our above arguments, $(A_1^\varphi, \chi_1^{sub}, \vartheta_1\lambda_1), \cdots, (A_n^\varphi, \chi_n^{sub}, \vartheta_n\lambda_n)$ are proof subsumptions. We can reconstruct $\chi$ from $\chi_1^{sub}, \cdots, \chi_n^{sub}$ and $\varphi$ from $A_1^\varphi, \ldots, A_n^\varphi$ using the definition of proof subsumption and the removed inference rules, and thus, $(\varphi, \chi, \vartheta\lambda)$ is a proof subsumption. $\square$

Proof subsumption, being based on multiset inclusion, fulfills the following property:

**Theorem 5.1.** *Let $(\varphi, \psi, \vartheta)$ be a proof subsumption and $\psi$ be an intuitionistic proof. Then $\varphi$ is intuitionistic as well.*

*Proof.* Since $(\varphi, \psi, \vartheta)$ is a proof subsumption, then for every sequent $S$ in $\varphi$, there exists a sequent $S'$ in $\psi$ such that $S\vartheta \subseteq S'$, for some substitution $\vartheta'$ (this property follows from the definition of proof subsumption). As $\psi$ is intuitionistic, all its sequents have at most one formula on the right side. So let $S\colon \Gamma \vdash \Delta$ be a sequent in $\varphi$; then there exists a sequent $S'\colon \Gamma' \vdash \Delta'$ in $\psi$ and a substitution $\vartheta$ such that $\Gamma\vartheta \subseteq \Gamma', \Delta\vartheta \subseteq \Delta'$. As $S'$ is intuitionistic $|\Delta'| \leq 1$, and so $|\Delta\vartheta| \leq 1$. As $\Delta$ is a multiset we have $|\Delta\vartheta| = |\Delta|$; so $\varphi$ is intuitionistic. $\square$

This result guarantees that the proof, eventually obtained by grounded proof resolution on the general (subsuming) level, is intuitionistic, provided that the proof obtained by grounded resolution of the $\mathrm{ACNF}^{\mathrm{top}}$'s projections is intuitionistic as well.

### *5.1. Projection Subsumption*

Using the subsumption principle for proofs we can relate the projections of two proofs that differ only by the application of some of Gentzen's reduction rules. In this section, we will show that, if $\varphi \rightsquigarrow_{\mathcal{R}^a} \varphi'$, then every projection of $\varphi'$ is subsumed by some projection of $\varphi$. This result is non-trivial, since the reduction steps might replace full subproofs even with different end-sequents.

**Definition 5.4 (Linearity).** A cut-free proof is $(\nu, C)$-linear if $\mathrm{es}(\varphi.\nu) = C \circ S$ and $C$ is passive below $\nu$, i.e. on the path $\pi(\nu)$ from $\nu$ to the end-sequent there are no inferences on the subsequent $C$. The path $\pi(\nu)$ is called a $C$-linear path in $\varphi$ from $\nu$.

Observe that if $C$ is the clause part of $\mathrm{es}(\varphi.\nu)$ on some node $\nu$ of a projection $\varphi$ to some clause, then $\varphi$ is $(\nu, C)$-linear by the definition of projections.

**Definition 5.5 (Dependency of nodes).** Let $\nu$ be a node in the proof $\varphi$. A node $\mu$ in $\varphi$ is called *$\nu$-dependent* if either $\mu$ occurs in $\varphi.\nu$ or $\mu$ occurs on the path from $\nu$ to the end-sequent. If $\mu$ is not $\nu$-dependent it is called *$\nu$-independent*.

Now we state two lemmas which describe proof transformations arising from projections when proofs are subjected to reductive cut-elimination. The first lemma is relevant for the cut-simplification of a universally quantified formula.

**Lemma 5.1.** *Let $\varphi$ be a skolemized cut-free proof, $\nu$ be a node in $\varphi$ such that $\mathrm{es}(\varphi.\nu) = C \circ S$ and $\varphi$ is $(\nu, C)$-linear. Let $(\varphi.\nu, \psi, \vartheta)$ be a proof subsumption such that $\mathrm{es}(\psi) = C\vartheta \circ S\vartheta$ and for all $\nu$-independent nodes $\mu$ in $\varphi$ we have $dom(\vartheta) \cap V(S(\mu)) = \emptyset$. We define $\varphi'$ as follows: replace $\varphi.\nu$ in $\varphi$ by $\psi$ and then replace the $(\nu, C)$-linear path from $\nu$ in $\varphi$ by a $(\nu, C\vartheta)$-linear path. Then $(\varphi, \varphi', \vartheta)$ is a proof subsumption.*

*Proof.* By induction on the length $l(\pi)$ of the $(\nu, C)$-linear path $\pi$.

**Base case:** $l(\pi) = 0$: trivial, as $\nu$ is the end node of $\varphi$. So $\varphi.\nu = \varphi$ and $\varphi' = \psi$.

**Induction hypothesis:** Assume that the lemma holds for all proofs such that for the $C$-linear path $\pi$ in $\varphi$ from $\nu$ we have $l(\pi) \leq n$.

**Inductive cases:** Assume $l(\pi) = n + 1$. Then $\pi = \pi'\nu_0$ where $\nu_0$ is the end node of $\varphi$. We distinguish two cases:

(a) The last inference in $\varphi$ is unary. Then $\varphi$ is of the form

$$\dfrac{\begin{matrix}(\varphi_1)\\ \mu\colon C_1, \Gamma' \vdash \Delta', C_2\end{matrix}}{\nu_0\colon C_1, \Gamma \vdash \Delta, C_2}\ \xi$$

where $\mu$ is the predecessor node of the end-sequent, $C = C_1 \vdash C_2$ and $\xi$ is an inference acting on $\Gamma' \vdash \Delta'$ (note that by $C$-linearity, there is no inference on the atoms in $C$). By definition, $\varphi'$ is of the form

$$\dfrac{\begin{matrix}(\varphi_1')\\ \mu'\colon C_1\vartheta, \Gamma'\vartheta \vdash \Delta'\vartheta, C_2\vartheta\end{matrix}}{\nu_0'\colon C_1\vartheta, \Gamma\vartheta \vdash \Delta\vartheta, C_2\vartheta}\ \xi$$

where $\xi$ has the same auxiliary formulas as in $\varphi$. By the induction hypothesis, $(\varphi_1, \varphi_1', \vartheta)$ is a proof subsumption and, by Definition 5.3 for unary inferences, $(\varphi, \varphi', \vartheta)$ is a proof subsumption as well.

(b) The last inference in $\varphi$ is binary. Without loss of generality, we assume that $\nu$ and the $C$-linear path are on the side of the left premise. Then $\varphi$ is of the form

$$\dfrac{\begin{matrix}(\varphi_1)\\ \mu_1\colon C_1, \Gamma_1' \vdash \Delta_1', C_2\end{matrix} \quad \begin{matrix}(\varphi_2)\\ \mu_2\colon \Gamma_2' \vdash \Delta_2'\end{matrix}}{\nu_0\colon C_1, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, C_2}\ \xi$$

where $C = C_1 \vdash C_2$ and $\xi$ is a binary inference acting on $\Gamma_1' \vdash \Delta_1'$ and $\Gamma_2' \vdash \Delta_2'$, where the principal formula of $\xi$ lies in $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$. Note that, for the case $\vee_l$, the binary rule is not purely multiplicative, but it is easy to see that the arguments below are the same. By definition, $\varphi'$ is of the form

$$\dfrac{\begin{matrix}(\varphi_1') & & (\varphi_2)\\ \mu_1' \colon C_1\vartheta, \Gamma_1'\vartheta \vdash \Delta_1'\vartheta, C_2\vartheta & & \mu_2 \colon \Gamma_2' \vdash \Delta_2'\end{matrix}}{\nu_0' \colon C_1\vartheta, \Gamma_1\vartheta, \Gamma_2 \vdash \Delta_1\vartheta, \Delta_2, C_2\vartheta}\ \xi$$

By induction hypothesis, $(\varphi_1,\ \varphi_1',\ \vartheta)$ is a proof subsumption. As $\mu_2$ is $\nu$-independent we have (by assumption on $\vartheta$) $V(\varphi_2) \cap \mathit{dom}(\vartheta) = \emptyset$ and thus $(\varphi_2, \varphi_2, \vartheta)$ is a proof subsumption. Then, by Definition 5.3 (binary case), $(\varphi, \varphi', \vartheta)$ is a proof subsumption as well. $\square$

The next lemma describes a typical case arising in projections when a cut-shifting rule is applied.

**Lemma 5.2.** *Let $\varphi$ be a skolemized cut-free proof, $\nu$ be a node in $\varphi$ such that* $\mathrm{es}(\varphi.\nu) = C \circ S$ *and $\varphi$ is $(\nu, C)$-linear. Let $(\varphi.\nu, \psi, \emptyset)$ be a proof subsumption such that* $\mathrm{es}(\psi) = C \circ D \circ S$ *for a sequent $D$. We define $\varphi'$ as follows: replace $\varphi.\nu$ in $\varphi$ by $\psi$ and replace the $C$-linear path $\pi(\nu)$ by a $C \circ D$-linear path. Then $(\varphi, \varphi', \emptyset)$ is a proof subsumption.*

*Proof.* By induction on the length $l(\pi)$ of the $(\nu, C)$-linear path $\pi$.

<u>**Base case:**</u> $l(\pi) = 0$: trivial as $\varphi = \varphi.\nu$ and $\varphi' = \psi$.

<u>**Induction hypothesis:**</u> Assume that the lemma holds for all proofs such that for the $C$-linear path $\pi$ in $\varphi$ from $\nu$ we have $l(\pi) \leq n$.

<u>**Inductive cases:**</u> Assume $l(\pi) = n + 1$. We distinguish two cases:

(a) The last inference in $\varphi$ is unary. Then $\varphi$ is of the form

$$\dfrac{\begin{matrix}(\varphi_1)\\ \mu \colon C_1, \Gamma' \vdash \Delta', C_2\end{matrix}}{\nu_0 \colon C_1, \Gamma \vdash \Delta, C_2}\ x$$

where $\mu$ is the predecessor node of the end-sequent, $C = C_1 \vdash C_2$ and $x$ is an inference acting on $\Gamma' \vdash \Delta'$. By definition, $\varphi'$ is of the form

$$\dfrac{\begin{matrix}(\varphi_1')\\ \mu' \colon C_1, D_1, \Gamma' \vdash \Delta', C_2, D_2\end{matrix}}{\nu_0' \colon C_1, D_1, \Gamma \vdash \Delta, C_2, D_2}\ x$$

For $D = D_1 \vdash D_2$. By the induction hypothesis, $(\varphi_1, \varphi_1', \emptyset)$ is a proof subsumption. By Definition 5.3 (unary case) $(\varphi, \varphi', \emptyset)$ is a proof subsumption as well.

(b) The last inference in $\varphi$ is binary. Without loss of generality, we assume that the $C$-linear path is on the side of the left premise. Then $\varphi$ is of the form

$$\dfrac{\begin{array}{c}(\varphi_1)\\ \mu_1 \colon C_1, \Gamma_1' \vdash \Delta_1', C_2\end{array} \quad \begin{array}{c}(\varphi_2)\\ \mu_2 \colon \Gamma_2' \vdash \Delta_2'\end{array}}{\nu_0 \colon C_1, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, C_2}\ x$$

where $C = C_1 \circ C_2$ and $x$ is a binary inference acting on $\Gamma_1' \vdash \Delta_1'$ and $\Gamma_2' \vdash \Delta_2'$, where the principal formula of $x$ lies in $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$. By definition, $\varphi'$ is of the form

$$\dfrac{\begin{array}{c}(\varphi_1)\\ \mu_1' \colon C_1, D_1, \Gamma_1' \vdash \Delta_1', C_2, D_2\end{array} \quad \begin{array}{c}(\varphi_2)\\ \mu_2 \colon \Gamma_2' \vdash \Delta_2'\end{array}}{\nu_0' \colon C_1, D_1, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, C_2, D_2}\ x$$

By the induction hypothesis, we know that $(\varphi_1, \varphi_1', \emptyset)$ is a proof subsumption. Obviously, $(\varphi_2, \varphi_2, \emptyset)$ is also a proof subumption. Then, by Definition 5.3 (binary case), $(\varphi, \varphi', \emptyset)$ is a proof subsumption as well. $\square$

The following lemma shows that for one-step proof reduction $\varphi \rightsquigarrow_{\mathcal{R}^a} \varphi'$, the projections of $\varphi'$ are "redundant" with respect to the projections of $\varphi$, in the sense that they are subsumed. It is worth noting that, although $\varphi$ and $\varphi'$ are **LJ**-proofs, their projections are possibly classical.

**Lemma 5.3** (Main Subsumption Lemma)**.** *Let $\varphi \rightsquigarrow_{\mathcal{R}^a} \varphi'$ and $\psi'$ be a projection for $\varphi'$. Then there exists a projection $\psi$ for $\varphi$ and a substitution $\vartheta$ such that $(\psi, \psi', \vartheta)$ is a proof subsumption.*

*Proof.* We show that the lemma holds for each of the rules in $\mathcal{R}^a$.

**Cut-elimination rule:**

Over weakening:

1. The cut-formula on the left branch is weakened. Then $\varphi$ contains a node $\nu$ such that:

$$\dfrac{\dfrac{\begin{array}{c}(\varphi_1)\\ \Gamma_1 \vdash\end{array}}{\Gamma_1 \vdash A}\ w_r \qquad \begin{array}{c}(\varphi_2)\\ A, \Gamma_2 \vdash \Delta\end{array}}{\Gamma_1, \Gamma_2 \vdash \Delta}\ cut$$

And $\varphi.\nu$ is reduced to $\chi$:

$$\dfrac{\begin{array}{c}(\varphi_1)\\ \Gamma_1 \vdash\end{array}}{\Gamma_1, \Gamma_2 \vdash \Delta}\ w_l^*, w_r^*$$

We define $\varphi' = \varphi[\chi]_\nu$. By definition 3.1, the characteristic clause sets are:

$$\mathrm{CL}(\varphi.\nu) = \mathrm{CL}(\varphi_1) \cup \mathrm{CL}(\varphi_2)$$
$$\mathrm{CL}(\chi) = \mathrm{CL}(\varphi_1)$$

This means that $\mathrm{CL}(\varphi') \subset \mathrm{CL}(\varphi)$ and consequently all projections of $\varphi'$ are also projections of $\varphi$, since no inference on end-sequent ancestors was modified[5]. Therefore, for every projection $\psi'$ of $\varphi'$, we take $\psi'$ itself as a projection of $\varphi$ and the proof subsumption $(\psi', \psi', \emptyset)$ holds.

2. The cut formula on the right branch is weakened: analogous to the previous case.

**Cut-shifting rules:**

Over unary inferences:

1. A unary inference $\rho$ is applied to the left branch of a cut on node $\nu$, so $\varphi.\nu$ is

$$\dfrac{\dfrac{\stackrel{(\varphi_1)}{\Gamma_1' \vdash A}}{\Gamma_1 \vdash A}\,\rho \qquad \stackrel{(\varphi_2)}{A, \Gamma_2 \vdash \Delta}}{\Gamma_1, \Gamma_2 \vdash \Delta}\; cut$$

Then $\varphi.\nu$ reduces to $\chi$:

$$\dfrac{\dfrac{\stackrel{(\varphi_1)}{\Gamma_1' \vdash A} \qquad \stackrel{(\varphi_2)}{A, \Gamma_2 \vdash \Delta}}{\Gamma_1', \Gamma_2 \vdash \Delta}\; cut}{\Gamma_1, \Gamma_2 \vdash \Delta}\,\rho$$

and we define $\varphi' = \varphi[\chi]_\nu$. Notice that $\mathrm{CL}(\varphi) = \mathrm{CL}(\varphi')$. We distinguish two cases:

(a) $\rho$ operates on a cut-ancestor. Then the rule shifting will not affect the projections and, since the clause sets are the same, so are the projections of $\varphi$ and $\varphi'$.

(b) $\rho$ operates on an end-sequent ancestor. Let $\psi'$ be a projection for $\varphi'.\nu$. Then $\psi'$ either goes over $\varphi_1$ or over $\varphi_2$. If $\psi'$ goes over $\varphi_1$ it is of the form

$$\dfrac{\dfrac{\stackrel{(\psi_1')}{\vec{C}, {}^*\Gamma_1' \vdash \vec{D}}}{\vec{C}, {}^*\Gamma_1', {}^*\Gamma_2 \vdash {}^*\Delta, \vec{D}}\; w_l^*, w_r^*}{\vec{C}, {}^*\Gamma_1, {}^*\Gamma_2 \vdash {}^*\Delta, \vec{D}}\;\rho$$

where ${}^*\Gamma_1$, ${}^*\Gamma_1'$, ${}^*\Gamma_2$ and ${}^*\Delta_1$ are suitable sub-multisets of $\Gamma_1$, $\Gamma_1'$, $\Gamma_2$ and $\Delta$, respectively, and $\vec{C} \vdash \vec{D}$ is a clause. The proof $\psi$:

$$\dfrac{\dfrac{\stackrel{(\psi_1')}{\vec{C}, {}^*\Gamma_1' \vdash \vec{D}}}{\vec{C}, {}^*\Gamma_1 \vdash \vec{D}}\;\rho}{\vec{C}, {}^*\Gamma_1, {}^*\Gamma_2 \vdash {}^*\Delta, \vec{D}}\; w_l^*, w_r^*$$

[5] Remember that projections are only composed by rules operating on end-sequent ancestors. See Definition 3.2.

is a projection for $\varphi.\nu$ which subsumes $\psi'$. Note that $\psi$ and $\psi'$ share the same end-sequent. So any projection for $\varphi'$ which is an extension of $\psi'$ is subsumed by a projection for $\varphi$.
If $\psi'$ goes over $\varphi_2$ it is of the form

$$\cfrac{\cfrac{\cfrac{(\tau)}{\vec{E}, {}^*\Gamma_2 \vdash {}^*\Delta, \vec{F}}}{\vec{E}, {}^*\Gamma_1, {}^*\Gamma_2 \vdash {}^*\Delta, \vec{F}}\ w_l^*}{\vec{E}, {}^*\Gamma_1, {}^*\Gamma_2 \vdash {}^*\Delta, \vec{F}}\ \rho$$

where $\vec{E} \vdash \vec{F}$ is a clause. Here, we define $\psi$:

$$\cfrac{\cfrac{(\tau)}{\vec{E}, {}^*\Gamma_2 \vdash {}^*\Delta, \vec{F}}}{\vec{E}, {}^*\Gamma_1, {}^*\Gamma_2 \vdash {}^*\Delta, \vec{F}}\ w_l^*$$

$\psi$ is a projection for $\varphi.\nu$ which subsumes $\psi'$. So any projection for $\varphi'$ which is an extension of $\psi'$ is subsumed by a projection for $\varphi$.

2. A unary inference $\rho$ is applied to the right branch: analogous to the previous case.

Over binary inferences:

1. Shifting the cut over the $\vee_l$-rule: Let $\varphi.\nu =$

$$\cfrac{\cfrac{\overset{(\varphi_1)}{P, \Gamma_1 \vdash A} \quad \overset{(\varphi_2)}{Q, \Gamma_2 \vdash A}}{P \vee Q, \Gamma_1, \Gamma_2 \vdash A}\ \vee_l \quad \overset{(\varphi_3)}{A, \Gamma_3 \vdash \Delta}}{P \vee Q, \Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\ cut$$

Then $\varphi.\nu$ reduces to $\chi$

$$\cfrac{\cfrac{\cfrac{\overset{(\varphi_1)}{P, \Gamma_1 \vdash A} \quad \overset{(\varphi_3)}{A, \Gamma_3 \vdash \Delta}}{P, \Gamma_1, \Gamma_3 \vdash \Delta}\ cut \quad \cfrac{\overset{(\varphi_2)}{Q, \Gamma_2 \vdash A} \quad \overset{(\varphi_3\sigma)}{A, \Gamma_3 \vdash \Delta}}{Q, \Gamma_2, \Gamma_3 \vdash \Delta}\ cut}{P \vee Q, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_3 \vdash \Delta}\ \vee_r}{P \vee Q, \Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\ c_l^*$$

where $\sigma$ is a renaming substitution for eigenvariables in $\varphi_3$ to keep the proof regular. As usual, we define $\varphi' = \varphi[\chi]_\nu$. In this case, the clause sets of the two proofs are different:

$$\begin{aligned}
\mathrm{CL}(\varphi.\nu) &= (\mathrm{CL}(\varphi_1) \star \mathrm{CL}(\varphi_2)) \cup \mathrm{CL}(\varphi_3) \\
\mathrm{CL}(\chi) &= (\mathrm{CL}(\varphi_1) \cup \mathrm{CL}(\varphi_3)) \star (\mathrm{CL}(\varphi_2) \cup \mathrm{CL}(\varphi_3\sigma))
\end{aligned}$$

where $\star \in \{\cup, \times\}$, depending on whether $P \vee Q$ is a cut or end-sequent ancestor. We distinguish two cases:

(a) $P \vee Q$ is an ancestor of a cut. In this case, we have:

$$\begin{aligned}
\mathrm{CL}(\varphi.\nu) &= \mathrm{CL}(\varphi_1) \cup \mathrm{CL}(\varphi_2) \cup \mathrm{CL}(\varphi_3) \\
\mathrm{CL}(\chi) &= \mathrm{CL}(\varphi_1) \cup \mathrm{CL}(\varphi_3) \cup \mathrm{CL}(\varphi_2) \cup \mathrm{CL}(\varphi_3\sigma)
\end{aligned}$$

This means that all projections of $\varphi$ are projections of $\varphi'$, and moreover $\varphi'$ contains projections that are simply different instantiations of $\varphi$'s projections. We analyze all these cases:

Consider a projection $\psi'$ for $\varphi'.\nu$. If $\psi'$ goes over $\varphi_1$, $\varphi_2$ or $\varphi_3$, then $\psi'$ is also a projection for $\varphi.\nu$ with the same end-sequent. In these cases the corresponding projection for $\varphi'$ is also a projection for $\varphi$.

If $\psi'$ goes over $\varphi_3\sigma$, then there is a projection $\psi$ for $\varphi.\nu$ such that $\psi\sigma = \psi'$ and $\psi$ subsumes $\psi'$. Now consider a projection $\tau'$ for $\varphi'$ which is an extension of $\psi'$. Then, by Lemma 5.1, we find a projection $\tau$ for $\varphi$ which subsumes $\tau'$.

(b) $P \vee Q$ is an end-sequent ancestor. In this case, the (normalized) clause sets are:

$$\begin{aligned}
\mathrm{CL}(\varphi.\nu) =& (\mathrm{CL}(\varphi_1) \times \mathrm{CL}(\varphi_2)) \cup \mathrm{CL}(\varphi_3) \\
\mathrm{CL}(\chi) =& (\mathrm{CL}(\varphi_1) \times \mathrm{CL}(\varphi_2)) \cup (\mathrm{CL}(\varphi_1) \times \mathrm{CL}(\varphi_3\sigma)) \\
& (\mathrm{CL}(\varphi_2) \times \mathrm{CL}(\varphi_3)) \cup (\mathrm{CL}(\varphi_3) \times \mathrm{CL}(\varphi_3\sigma))
\end{aligned}$$

Here, there are several cases to analyze:

i. $\psi'$ is a projection for $\chi$ which goes over $\varphi_1$ and $\varphi_2$. This is the simplest case as $\psi'$ is also a projection for $\varphi.\nu$ with the same end-sequent. The extensions to the projections for $\varphi'$ and $\varphi$ are as usual.

ii. $\psi'$ is a projection for $\chi$ which goes over $\varphi_1$ and $\varphi_3\sigma$. Then $\psi'$ is:

$$\dfrac{\dfrac{(\psi_1')}{\vec{C}, P, {}^*\Gamma_1 \vdash \vec{D}} \quad \dfrac{\dfrac{(\psi_3'\sigma)}{\vec{E}\sigma, {}^*\Gamma_3\sigma \vdash {}^*\Delta\sigma, \vec{F}\sigma}}{Q\sigma, \vec{E}\sigma, {}^*\Gamma_3\sigma \vdash {}^*\Delta\sigma, \vec{F}\sigma}\ w_l^*}{\vec{C}, \vec{E}\sigma, P \vee Q\sigma, {}^*\Gamma_1, {}^*\Gamma_3\sigma \vdash {}^*\Delta\sigma, \vec{D}, \vec{F}\sigma}\ \vee_l$$

Then the following projection $\psi$ for $\varphi.\nu$ subsumes $\psi'$ (more precisely $(\psi, \psi', \sigma)$ is a proof subsumption):

$$\dfrac{\dfrac{(\psi_3')}{\vec{E}, {}^*\Gamma_3 \vdash {}^*\Delta, \vec{F}}}{\vec{E}, P \vee Q, {}^*\Gamma_1, {}^*\Gamma_3 \vdash {}^*\Delta, \vec{F}}\ w_l^*$$

Note that $dom(\sigma) \cap V(P, {}^*\Gamma_1) = \emptyset$ as $dom(\sigma) \cap V(\varphi_1) = \emptyset$ and so $\psi$ subsumes $\psi\sigma$ which in turn subsumes $\psi'$. We apply both Lemma 5.1 and Lemma 5.2 to extend the projections to $\varphi$.

iii. $\psi'$ is a projection for $\chi$ which goes over $\varphi_2$ and $\varphi_3\sigma$: analogous to 1(b)ii.

iv. $\psi'$ is a projection for $\chi$ which goes over $\varphi_3$ and $\varphi_3\sigma$: analogous to 1(b)ii.

2. Shifting the cut over $\to_l$: Let $\varphi.\nu$:

$$\dfrac{\dfrac{\begin{matrix}(\varphi_1)\\ \Gamma_1 \vdash P\end{matrix} \quad \begin{matrix}(\varphi_2)\\ Q, \Gamma_2 \vdash A\end{matrix}}{P \to Q, \Gamma_1, \Gamma_2 \vdash A}\,\to_l \qquad \begin{matrix}(\varphi_3)\\ A, \Gamma_3 \vdash \Delta\end{matrix}}{P \to Q, \Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\,cut$$

Which is transformed to $\chi$:

$$\dfrac{\begin{matrix}(\varphi_1)\\ \Gamma_1 \vdash P\end{matrix} \qquad \dfrac{\begin{matrix}(\varphi_2)\\ Q, \Gamma_2 \vdash A\end{matrix} \quad \begin{matrix}(\varphi_3)\\ A, \Gamma_3 \vdash \Delta\end{matrix}}{Q, \Gamma_2, \Gamma_3 \vdash \Delta}\,cut}{P \to Q, \Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\,\to_l$$

We define $\varphi'$ as $\varphi[\chi]_\nu$. The clause sets are:

$$\begin{aligned}\mathrm{CL}(\varphi.\nu) &= (\mathrm{CL}(\varphi_1) \star \mathrm{CL}(\varphi_2)) \cup \mathrm{CL}(\varphi_3)\\ \mathrm{CL}(\chi) &= \mathrm{CL}(\varphi_1) \star (\mathrm{CL}(\varphi_2) \star \mathrm{CL}(\varphi_3))\end{aligned}$$

where $\star \in \{\cup, \times\}$, depending on whether $P \to Q$ is a cut or end-sequent ancestor. As before, we analyze both cases:

(a) $P \to Q$ is a cut ancestor. In this case, we have:

$$\begin{aligned}\mathrm{CL}(\varphi.\nu) &= \mathrm{CL}(\varphi_1) \cup \mathrm{CL}(\varphi_2) \cup \mathrm{CL}(\varphi_3)\\ \mathrm{CL}(\chi) &= \mathrm{CL}(\varphi_1) \cup \mathrm{CL}(\varphi_2) \cup \mathrm{CL}(\varphi_3)\end{aligned}$$

Therefore, the clause sets and consequently projections of $\varphi$ and $\varphi'$ are the same. The theorem holds trivially.

(b) $P \to Q$ is an end-sequent ancestor. The normalized clause sets are:

$$\begin{aligned}\mathrm{CL}(\varphi.\nu_{)} &= (\mathrm{CL}(\varphi_1) \times \mathrm{CL}(\varphi_2)) \cup \mathrm{CL}(\varphi_3)\\ \mathrm{CL}(\chi) &= (\mathrm{CL}(\varphi_1) \times \mathrm{CL}(\varphi_2)) \cup (\mathrm{CL}(\varphi_1) \times \mathrm{CL}(\varphi_3))\end{aligned}$$

We need to analyze two cases:

i. $\psi'$ is a projection for $\chi$ that goes over $\varphi_1$ and $\varphi_2$. Analogous to case 1(b)i, where $\psi'$ is also a projection for $\varphi.\nu$.

ii. $\psi'$ is a projection for $\chi$ that goes over $\varphi_1$ and $\varphi_3$. Analogous to case 1(b)ii, where a projection of $\varphi.\nu$ going over $\varphi_3$ subsumes $\psi'$.

3. Shifting the cut over other binary rules: analogous to the previous case.

**Cut-simplification rules:**

1. Of a cut on an $\wedge$-formula: Let us assume that $\varphi.\nu$ is a cut-derivation of the form

$$\cfrac{\cfrac{\begin{array}{cc}(\varphi_1) & (\varphi_2)\\ \Gamma_1 \vdash A & \Gamma_2 \vdash B\end{array}}{\Gamma_1, \Gamma_2 \vdash A \wedge B}\ \wedge_r \qquad \cfrac{\begin{array}{c}(\varphi_3)\\ A, \Gamma_3 \vdash \Delta\end{array}}{A \wedge B, \Gamma_3 \vdash \Delta}\ \wedge_l}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\ cut$$

   Then $\varphi.\nu$ reduces to $\chi$:

$$\cfrac{\cfrac{\begin{array}{cc}(\varphi_1) & (\varphi_3)\\ \Gamma_1 \vdash A & A, \Gamma_3 \vdash \Delta\end{array}}{\Gamma_1, \Gamma_3 \vdash \Delta}\ cut}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta}\ w_l^*$$

   And we define $\varphi' = \varphi[\chi]_\nu$. It is easy to see that every projection of $\varphi'$ is also a projection of $\varphi$ (due to the fact that $\mathrm{CL}(\varphi') \subseteq \mathrm{CL}(\varphi)$). Indeed, if the projection $\psi'$ goes over $\varphi_1$ and weakens material from $\varphi_3$, then we take the same projection $\psi'$ in $\varphi$, "ignore" $\varphi_2$ and do the same weakenings as in $\varphi'$.

2. Of a cut on an $\vee$- and $\rightarrow$-formula: analogous to the $\wedge$ case.

3. Of a cut on an $\neg$-formula: Let us assume that $\varphi.\nu$ is a cut-derivation of the form

$$\cfrac{\cfrac{\begin{array}{c}(\varphi_1)\\ A, \Gamma_1 \vdash\end{array}}{\Gamma_1 \vdash \neg A}\ \neg_r \qquad \cfrac{\begin{array}{c}(\varphi_2)\\ \Gamma_2 \vdash A\end{array}}{\neg A, \Gamma_2 \vdash}\ \neg_l}{\Gamma_1, \Gamma_2 \vdash}\ cut$$

   Then $\varphi.\nu$ reduces to $\chi$

$$\cfrac{\begin{array}{cc}(\varphi_2) & (\varphi_1)\\ \Gamma_2 \vdash A & A, \Gamma_1 \vdash\end{array}}{\Gamma_1, \Gamma_2 \vdash}\ cut$$

   Now consider the proof $\varphi'$ as $\varphi[\chi]_\nu$ and a projection $\psi'$ of $\varphi'$. It is easy to see that $\psi'$ is also a projection of $\varphi$: either the projection does not pass over the node $\nu$ (in which case they are trivially equal), or the projection goes over $\nu$. But the inferences $\neg_l$ and $\neg_r$ are unary inferences going into a cut and thus are dropped in the projections. Therefore also the projections going over $\nu$ are the same in $\varphi$ and $\varphi'$. In particular we have $\mathrm{CL}(\varphi) = \mathrm{CL}(\varphi')$.

4. Of a cut on an $\exists$-formula: analogous to the $\forall$ case which follows.

5. Of a cut on a $\forall$-formula: Let us assume that $\varphi.\nu$ is a cut-derivation of the form

$$\cfrac{\cfrac{\begin{array}{c}(\varphi_1(\alpha))\\ \Gamma_1 \vdash A(\alpha)\end{array}}{\Gamma_1 \vdash \forall x.A(x)}\ \forall_r \qquad \cfrac{\begin{array}{c}(\varphi_2)\\ A(t), \Gamma_2 \vdash \Delta\end{array}}{\forall x.A(x), \Gamma_2 \vdash \Delta}\ \forall_l}{\Gamma_1, \Gamma_2 \vdash \Delta}\ cut$$

Then $\varphi.\nu$ reduces to $\chi$

$$\dfrac{\begin{matrix}(\varphi_1(t)) \\ \Gamma_1 \vdash A(t)\end{matrix} \quad \begin{matrix}(\varphi_2) \\ A(t), \Gamma_2 \vdash \Delta\end{matrix}}{\Gamma_1, \Gamma_2 \vdash \Delta}\ cut$$

and $\varphi' = \varphi[\chi]_\nu$.
Now, let $\psi'$ be a projection of $\varphi'$. If the projection does not go over $\nu$ then it is also a projection of $\varphi$. If the projection goes over $\nu$ in $\varphi'$ by coming from $\varphi_2$ (and using weakenings of formulas from $\Gamma_1$), then this projection is also a projection in $\varphi$. The interesting case is when the projection $\psi'$ comes from $\varphi_1(t)$. We consider $\psi'.\nu$ and the corresponding projection $\psi.\nu$ in $\varphi$. Now, as $\psi$ is a projection, there exists a subsequent $C$ of $\mathrm{es}(\psi.\nu)$ and $\psi$ is $(\nu, C)$-linear. Let $\vartheta = \{\alpha \leftarrow t\}$. First of all, it is easy to see (use the definition of proof subsumption) that $(\psi.\nu, \psi'.\nu, \vartheta)$ is a proof subsumption and that (due to the regularity of $\varphi$) the conditions in Lemma 5.1 are fulfilled and the corresponding proof $\varphi'$ (as defined in the lemma) is exactly the projection $\psi'$. So $(\psi, \psi', \vartheta)$ is a proof subsumption.

6. Of a cut on a contracted formula: Let $\varphi.\nu =$

$$\dfrac{\begin{matrix}(\varphi_1) \\ \Gamma_1 \vdash A\end{matrix} \quad \dfrac{\begin{matrix}(\varphi_2) \\ A, A, \Gamma_2 \vdash \Delta\end{matrix}}{A, \Gamma_2 \vdash \Delta}\ c_l}{\Gamma_1, \Gamma_2 \vdash \Delta}\ cut$$

Then $\varphi.\nu$ reduces to $\chi$:

$$\dfrac{\dfrac{\begin{matrix}(\varphi_1) \\ \Gamma_1 \vdash A\end{matrix} \quad \dfrac{\begin{matrix}(\varphi_1\sigma) \\ \Gamma_1 \vdash A\end{matrix} \quad \begin{matrix}(\varphi_2) \\ A, A, \Gamma_2 \vdash \Delta\end{matrix}}{A, \Gamma_1, \Gamma_2 \vdash \Delta}\ cut}{\Gamma_1, \Gamma_1, \Gamma_2 \Delta}\ cut}{\Gamma_1, \Gamma_2 \vdash \Delta}\ c_l^*$$

Now, let $\varphi' = \varphi[\chi]_\nu$ and let $\psi'$ be a projection for $\varphi'.\nu$. If $\psi'$ goes over $\varphi_1$ or $\varphi_2$ then $\psi'$ is also a projection for $\varphi.\nu$. If $\psi'$ goes over $\varphi_1\sigma$ (where $\sigma$ is a renaming of eigenvariables in $\varphi_1$) then there exists a projection $\psi$ for $\varphi.\nu$ going over $\varphi_1$ such that $\psi\sigma = \psi'$ and $(\psi, \psi', \sigma)$ is a proof subsumption. The extension to the corresponding projection for $\varphi$ is done via Lemma 5.1.

□

**Theorem 5.2.** *Let $\varphi \rightsquigarrow^*_{\mathcal{R}^a} \varphi'$ and $\psi'$ be a projection for $\varphi'$. Then there exists a projection $\psi$ for $\varphi$ and a substitution $\vartheta$ such that $(\psi, \psi', \vartheta)$ is a proof subsumption.*

*Proof.* By Lemma 5.3 and transitivity of proof subsumption (Proposition 5.1).
□

**Corollary 5.1.** *Let $\varphi'$ be an* ACNF *of $\varphi$ obtained under $\mathcal{R}^a$. Then for any projection $\psi'$ of $\varphi'$ there exists a projection $\psi$ of $\varphi$ and a substitution $\vartheta$ such that $(\psi, \psi', \vartheta)$ is a proof subsumption.*

*Proof.* By Theorem 5.2. □

## 6. Joining Proofs by Proof Resolution

As we have already noted in Section 3.1, the implementation of a specially designed resolution refinement is not enough for extending `CERES` to intuitionistic logic. One issue that cannot be dealt with easily, is the repetition of end-sequent formulas on the right-hand side of the sequent when computing a *context product* (see Definition 3.4). This is expected given the classical nature of the resolution procedure. One can see how this can occur, especially in first-order logic, when a single projection is used multiple times with different variable instantiations. Of course, it is not enough to introduce contractions, given that the original reason for the structural constraints found in the intuitionistic calculus was to limit the use of contraction in the first place.

Thus, we are left with the need to choose which of the repeated end-sequent formulas stay in the proof and which do not. Resolving of cut-free proofs makes this procedure tractable by allowing every possible way in which the cut-free proofs can be combined. Proof resolution, which is based on grounded proof resolution, is a way of introducing implicit contractions inferences without actually introducing contraction rules in the proof itself; the contraction is done while resolving the two proofs together. Note that we need a definition on the general and not only on the ground level. We are resolving cut-free proofs via atoms and most general unification, i.e., we have cut-free proofs $\varphi$ of $\Gamma \vdash \Delta, A$ and $\varphi'$ of $B, \Pi \vdash \Lambda$ such that $A$ and $B$ are unifiable atoms with most general unifier (m.g.u.) $\sigma$. We then consider the proofs $\varphi\sigma$ and $\varphi'\sigma$ and combine them via atomic cut. The stepwise shifting of this cut is needed to define proof resolution inductively.

**Definition 6.1 (Ground proof resolution).** Let $\varphi$ and $\psi$ be cut-free proofs with end sequents $\Gamma \vdash \Delta, A^m$ and $A^n, \Pi \vdash \Lambda$, where $A$ is atomic and $n, m \geq 1$. The grounded resolution of $\varphi$ and $\psi$ on $(A^m, A^n)$ is a set of cut-free proofs denoted by $\varphi \bowtie \psi$. Resolutions can be left- or right-resolutions, i.e. $\varphi \bowtie \psi = \varphi \bowtie_l \psi \cup \varphi \bowtie_r \psi$. We define $\varphi \bowtie_l \psi$ and $\varphi \bowtie_r \psi$ inductively as follows:

<u>**Base case:**</u> When $\varphi = A \vdash A$ and $\psi = A \vdash A$ then $\varphi \bowtie_l \psi = \varphi \bowtie_r \psi = \{A \vdash A\}$.

<u>**Inductive cases:**</u> By $\varphi \bowtie_l \psi$ we refer to the ground proof resolution of $\varphi$ and $\psi$ by considering the left branch rule before the right branch rule. By $\varphi \bowtie_r \psi$ we refer to the opposite. We will only define $\varphi \bowtie_l \psi$. The definition of $\varphi \bowtie_r \psi$ is symmetric. Since $A$ is atomic, no logical rules are applied to it.

1. Weakening rule on $A$:

If $m = 1$ then $\varphi = \cfrac{\begin{matrix}(\varphi')\\ \Gamma \vdash \Delta\end{matrix}}{\Gamma \vdash \Delta, A}\; w_r$ . In this case,

$$\varphi \bowtie_l \psi = \left\{ \frac{\varphi'}{S}\; w* \;\middle|\; S \subseteq \Gamma, \Pi \vdash \Delta, \Lambda \right\}.$$

If $m > 1$, then $\varphi = \cfrac{\begin{matrix}(\varphi')\\ \Gamma \vdash \Delta, A^m\end{matrix}}{\Gamma \vdash \Delta, A^{m+1}}\; w_r$ . In this case, $\varphi \;\bowtie_l\; \psi = \varphi' \;\bowtie\; \psi$.

2. Contraction rule on $A$:
   Let $\varphi = \cfrac{\begin{matrix}(\varphi')\\ \Gamma \vdash \Delta, A^{m+1}\end{matrix}}{\Gamma \vdash \Delta, A^m}\; c_r$ then $\varphi \;\bowtie_l\; \psi = \varphi' \;\bowtie\; \psi$.

3. Unary logical rule
   Let $\varphi = \cfrac{\begin{matrix}(\varphi')\\ \Gamma' \vdash \Delta', A^m\end{matrix}}{\Gamma \vdash \Delta, A^m}\; x$ then

$$\varphi \;\bowtie_l\; \psi = \left\{ \frac{\chi}{\overline{\overline{S}}}\; w^*, x \;\middle|\; \chi \in \varphi' \;\bowtie\; \psi \;\wedge\; S \subseteq \Gamma, \Pi \vdash \Delta, \Lambda \right\}$$

   Where the principal formula of $x$ is possibly weakened.

4. Binary logical rule
   Unlike the previous cases, there is no general form for binary rules. Thus, we show only the case for $\wedge_r$ as an example. The other rules behave in a similar way, except for the location of the auxiliary formulas within the sequent.
   Let $\varphi = \cfrac{\begin{matrix}(\varphi_1)\\ \Gamma_1 \vdash \Delta_1, P, A^k\end{matrix} \quad \begin{matrix}(\varphi_2)\\ \Gamma_2 \vdash \Delta_2, Q, A^l\end{matrix}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, P \wedge Q, A^{k+l}}\; \wedge_r$ , then

$$\varphi \;\bowtie_l\; \psi = \left\{ \begin{matrix} \dfrac{\chi_1 \quad \chi_2}{S'}\; \wedge_r \\ \overline{\overline{S}}\; c^*, w^* \end{matrix} \;\middle|\; \begin{matrix} \chi_1 \in \varphi_1 \;\bowtie\; \psi \;\wedge \\ \chi_2 \in \varphi_2 \;\bowtie\; \psi \;\wedge \\ S \subseteq \Gamma_1, \Gamma_2, \Pi \vdash \Delta_1, \Delta_2, \Lambda \end{matrix} \right\}$$

   This case holds iff the end sequent of $\varphi_1 \;\bowtie\; \psi$ and $\varphi_2 \;\bowtie\; \psi$ contain $P$ and $Q$, respectively. If one of the formulas is missing, we add a weakening rule after the respective resolution. If neither formula occurs, then the resolution is:

$$\varphi \;\bowtie_l\; \psi = \left\{ \frac{\chi}{\overline{\overline{S}}}\; w^* \;\middle|\; \chi \in \varphi_1 \bowtie \psi \cup \varphi_2 \bowtie \psi,\; S \subseteq \Gamma_1, \Gamma_2, \Pi \vdash \Delta_1, \Delta_2, \Lambda \right\}$$

Now, we introduce a (non-ground) resolution principle for cut-free proofs. While resolution of two projections is not necessarily a projection (to the same set of clauses), the resolution of cut-free proofs is again a cut-free proof.

**Definition 6.2 (Proof resolution).** Let $\varphi$ be a cut-free proof of $\Gamma \vdash \Delta, A_1, \ldots, A_n$ and $\psi$ be a cut-free proof of $B_1, \ldots, B_m, \Pi \vdash \Lambda$. Let $\sigma$ be a most general unifier of $\{A_1, \ldots, A_n, B_1, \ldots, B_m\}$. Then any ground resolution $\chi \in \varphi\sigma \bowtie \psi\sigma$ on $(A\sigma^n, B\sigma^m)$ is called a *resolvent* of $\varphi$ and $\psi$ on $(A_1, \ldots, A_n, B_1, \ldots, B_m)$. We write $\varphi \circ_r \psi$ for $\varphi\sigma \bowtie \psi\sigma$.

Note that, in the above definition, we require simultaneous unification of several atoms (as it is required in factoring of resolution).

**Example 6.1.** Let $\varphi$ be the following proof with a cut in which the cut-formulas are labeled with an index:

$$\dfrac{\dfrac{\dfrac{\dfrac{Py \vdash Py^1 \quad Py \vdash Py^1}{Py \vee Py \vdash Py^1}\,\vee_l}{\forall x.(Px \vee Px) \vdash Py^1}\,\forall_l}{\forall x.(Px \vee Px) \vdash \forall x.Px^1}\,\forall_r \qquad \dfrac{\dfrac{\dfrac{Pa^1 \vdash Pa}{\forall x.Px^1 \vdash Pa}\,\forall_l \quad \dfrac{Pa^1 \vdash Pa}{\forall x.Px^1 \vdash Pa}\,\forall_l}{\forall x.Px^1, \forall x.Px^1 \vdash Pa \wedge Pa}\,\wedge_r}{\forall x.Px^1 \vdash Pa \wedge Pa}\,c_l}{\forall x.(Px \vee Px) \vdash Pa \wedge Pa}\,cut$$

Its characteristic clause set is:

$$\mathrm{CL}(\varphi) = \{\vdash Py^1, Py^1 \;\; ; \;\; Pa^1, Pa^1 \vdash\}$$

And the minimal projections are $\pi_1$:

$$\dfrac{\dfrac{Py \vdash Py^1 \quad Py \vdash Py^1}{Py \vee Py \vdash Py^1, Py^1}\,\vee_l}{\forall x.(Px \vee Px) \vdash Py^1, Py^1}\,\forall_l$$

and $\pi_2$:

$$\dfrac{Pa^1 \vdash Pa \quad Pa^1 \vdash Pa}{Pa^1, Pa^1 \vdash Pa \wedge Pa}\,\wedge_r$$

Note that, as expected, the projections are classical. The projections $\pi_1$ and $\pi_2$ contain double occurrences of atoms, so we have to resolve on $(Py, Py, Pa, Pa)$.

A resolvent of $\pi_1$ and $\pi_2$ is a proof in $\pi_1\vartheta \bowtie \pi_2\vartheta$ for $\vartheta = \{y \leftarrow a\}$. We construct a proof $\chi_1$ in $\pi_1\vartheta \bowtie \pi_2\vartheta$. Let $\chi_1$ be:

$$\dfrac{\dfrac{(\chi_2) \quad (\chi_2)}{Pa \vee Pa \vdash Pa \wedge Pa}\,\vee_l}{\forall x.(Px \vee Px) \vdash Pa \wedge Pa}\,\forall_l$$

For $\chi_2 \in Pa \vdash Pa^1 \bowtie \pi_2$. It remains to define $\chi_2$:

$$\dfrac{\dfrac{(\chi_3) \quad (\chi_3)}{Pa, Pa \vdash Pa \wedge Pa}\,\wedge_r}{Pa \vdash Pa \wedge Pa}\,c_l$$

for $\chi_3 \in Pa \vdash Pa^1 \bowtie Pa^1 \vdash Pa$. But $Pa \vdash Pa^1 \bowtie Pa^1 \vdash Pa = \{Pa \vdash Pa\}$ and $\chi_3 = Pa \vdash Pa$. It is easy to see that $\chi_1 \in \pi_1 \circ_r \pi_2$ is an intuitionistic proof which can be obtained via cut-elimination on $\varphi$.

Note that proof resolution has an interesting side-effect. Resolving projections with end-sequents $C\sigma, \Gamma\sigma \vdash \Delta\sigma, D\sigma, A\sigma$ and $A\sigma, E\sigma, \Pi\sigma \vdash \Lambda\sigma, F\sigma$ in general gives a proof of $C'\sigma, E'\sigma, \Gamma'\sigma, \Pi'\sigma \vdash \Delta'\sigma, \Lambda'\sigma, D'\sigma, F'\sigma$, where $C', E' \vdash D', F'$ is (possibly a proper) subsequent of $C, E \vdash D, F$ (due to redundant branches where the atoms to be resolved do not occur). So the clause part of the proof resolution does not always coincide with a resolvent $R$ of the clauses $C \vdash D, A$ and $B, E \vdash F$, but it subsumes $R$.

**Lemma 6.1.** *Let $\varphi, \psi$ be cut-free proofs and $(\varphi, \psi, \vartheta)$ a proof subsumption such that $\mathrm{es}(\psi) = \Gamma \vdash \Delta, A^m$ for an atom $A$ and $m \geq 1$, and $\mathrm{es}(\varphi)\vartheta \subseteq \Gamma \vdash \Delta$. Let $\chi$ be an $A$-resolution of $\psi$ and a cut-free proof $\rho$. Then $(\varphi, \chi, \vartheta)$ is a proof subsumption.*

*Proof.* It is easy to show that this subsumption relation is invariant under ground resolution rules. The formal proof is by induction on the heights of $\psi$ and $\rho$. We consider only the more complicated cases. Let $\rho$ be a proof of $A^k, \Pi \vdash \Lambda$.

**<u>Base case:</u>** Both $\psi$ and $\rho$ have height 1. Then they are both:

$$\overline{A \vdash A}$$

and the proof resolution $\chi$ is $\overline{A \vdash A}$. In this case, $\mathrm{es}(\varphi)\vartheta \subseteq \Gamma \vdash \Delta$ would mean $\mathrm{es}(\varphi) = A' \vdash$ for an atom $A'$ which is impossible as $A' \vdash$ is not provable in **LK**. So the theorem is trivially true (if $A' \vdash$ were provable, then $\varphi$ would subsume $\chi$ anyway).

**<u>Inductive cases:</u>** We give a proof for a left-resolution via $\wedge_r$; all other cases are similar.

1. Let $\Gamma \vdash \Delta = \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, B \wedge C$ and $\chi$ be a (left-) resolution of

$$\dfrac{\overset{(\psi_1)}{\Gamma_1 \vdash \Delta_1, B, A^{m_1}} \quad \overset{(\psi_2)}{\Gamma_2 \vdash \Delta_2, C, A^{m_2}}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, B \wedge C, A^m}\ \wedge_r \quad \text{and} \quad \overset{(\rho)}{A^k, \Pi \vdash \Lambda}$$

Then, by definition, $\chi$ is of the form

$$\dfrac{\dfrac{(\chi_1) \quad (\chi_2)}{S'}\ \wedge_r}{S}\ s^*$$

where

$$\begin{array}{rcl} \chi_1 & \in & \overset{(\psi_1)}{\Gamma_1 \vdash \Delta_1, B, A^{m_1}} \bowtie \overset{(\rho)}{A^k, \Pi \vdash \Lambda}, \\ \chi_2 & \in & \overset{(\psi_2)}{\Gamma_2 \vdash \Delta_2, C, A^{m_2}} \bowtie \overset{(\rho)}{A^k, \Pi \vdash \Lambda}. \end{array}$$

we distinguish 3 cases:

(a) $\mathrm{es}(\varphi)\vartheta \subseteq \Gamma_1 \vdash \Delta_1$. Then we can apply the induction hypothesis and $(\varphi, \chi_1, \vartheta)$ is a proof subsumption. Then by definition of proof subsumption also $(\varphi, \chi, \vartheta)$ is a proof subsumption.

(b) 1a does not hold but $\mathrm{es}(\varphi)\vartheta \subseteq \Gamma_2 \vdash \Delta_2$. Analogous to 1a and $(\varphi, \chi_2, \vartheta)$ is a proof subsumption; by definition of proof subsumption $(\varphi, \chi, \vartheta)$ is a proof subsumption as well.

(c) Neither 1a nor 1b holds. Then by the conditions on $\varphi$ and by the definition of proof subsumption $\varphi$ is of the form

$$\cfrac{\begin{array}{cc}(\varphi_1) & (\varphi_2)\\ \Gamma_1' \vdash \Delta_1', B' & \Gamma_2' \vdash \Delta_2', C'\end{array}}{\Gamma_1', \Gamma_2' \vdash \Delta_1', \Delta_2', B' \wedge C'}\ \wedge_r$$

such that $(\varphi_1, \psi_1, \vartheta)$ and $(\varphi_2, \psi_2, \vartheta)$ are proof subsumptions. By induction hypothesis $(\varphi_1, \chi_1, \vartheta)$ and $(\varphi_2, \chi_2, \vartheta)$ are also proof subsumptions. By definition of proof subsumption, we get that $(\varphi, \chi, \vartheta)$ is a proof subsumption as well.

2. All cases of right-resolution (via the proof $\rho$) are simpler.

□

**Lemma 6.2.** *Let $\varphi, \psi$ be cut-free proofs and $(\varphi, \psi, \vartheta)$ a proof subsumption such that $\mathrm{es}(\psi) = A^m, \Gamma \vdash \Delta$ for an atom $A$ and $m \geq 1$, and $\mathrm{es}(\varphi)\vartheta \subseteq \Gamma \vdash \Delta$. Let $\chi$ be an $A$-resolution of $\psi$ and a cut-free proof $\rho$. Then $(\varphi, \chi, \vartheta)$ is a proof subsumption.*

*Proof.* As for Lemma 6.1. □

**Lemma 6.3.** *Let $(\varphi, \varphi', \eta)$ and $(\psi, \psi', \eta)$ be proof subsumptions among cut-free proofs such that $\varphi$ is a proof of $\Gamma \vdash \Delta, A^m$ and $\psi$ is a proof of $A^k, \Pi \vdash \Lambda$ where $A$ is an atom and $m, k \geq 1$. Let $\mathrm{es}(\varphi') = \Gamma' \vdash \Delta', (A\eta)^p$ and $\mathrm{es}(\psi') = (A\eta)^q, \Pi' \vdash \Lambda'$ such that $\Gamma\eta \subseteq \Gamma'$, $\Delta\eta \subseteq \Delta'$, $\Pi\eta \subseteq \Pi'$, $\Lambda\eta \subseteq \Lambda'$ and $p \geq m, q \geq k$. Let $\chi'$ be a ground proof resolution of $\varphi'$ and $\psi'$ on $(A\eta^p, A\eta^q)$. Then there exists a ground proof resolution $\chi$ of $\varphi$ and $\psi$ on $(A^m, A^k)$ such that $(\chi, \chi', \eta)$ is a proof subsumption.*

*Proof.* We can assume that $\mathrm{es}(\varphi') = \Gamma' \vdash \Delta', B^p$, $\mathrm{es}(\psi') = B^q, \Pi' \vdash \Lambda'$ for $B = A\eta$. We consider a $\chi' \in \varphi' \bowtie \psi'$ on $(B^p, B^q)$. The proof is by induction on the heights of $\varphi'$ and $\psi'$.

<u>**Base case:**</u> Both $\varphi'$ and $\psi'$ have height 1. Then they are both

$$\overline{B \vdash B}$$

The only ground proof resolution of $\varphi'$ and $\psi'$ is $\overline{B \vdash B}$. In this case $\varphi = \overline{A \vdash A}$ and $\psi = \overline{A \vdash A}$ and their ground proof resolution is $\overline{A \vdash A}$. Clearly $(\overline{A \vdash A}, \overline{B \vdash B}, \eta)$ is a proof subsumption.

**Induction hypothesis:** Assume the lemma holds for $\varphi'$ and $\psi'$ with heights $h$ and $i$ such that $h + i \leq n$.

**Inductive cases:** We consider only the ground proof resolutions via rank reduction in $\varphi'$ (left-resolution), the other cases are analogous. In particular, we assume that the number of nodes in $\varphi'$ is greater than 1.

- The last rule in $\varphi'$ is a unary rule $x$.

  We distinguish two cases:

  1. The principal formula of $x$ is not in $B^p$: then $\varphi' =$

$$\cfrac{\begin{matrix}(\varphi_1')\\ \Gamma' \vdash \Delta', B^p\end{matrix}}{\Gamma'' \vdash \Delta'', B^p}\ x$$

     Let $\chi' \in \varphi' \bowtie \psi'$ via $x$. Then there exists a $\chi_1' \in \varphi_1' \bowtie \psi'$ such that $\chi' =$

$$\cfrac{\cfrac{\begin{matrix}(\chi_1')\\ \Gamma^*, \Pi^* \vdash \Delta^*, \Lambda^*\end{matrix}}{\Gamma_1^*, \Pi^* \vdash \Delta_1^*, \Lambda^*}\ x}{S}\ s^*$$

     where $\Gamma^*, \Pi^* \vdash \Delta^*, \Lambda^*$ is a subsequent of $\Gamma', \Pi' \vdash \Delta', \Lambda'$ and the auxiliary formula of $x$ is in $\Gamma^* \vdash \Delta^*$. The inferences $s^*$ are a sequence of constractions followed by a sequence of weakenings. $S$ is a subsequent of $\Gamma'', \Pi' \vdash \Delta'', \Lambda'$. If the auxiliary formula is not in $\Gamma^* \vdash \Delta^*$, then $\chi_1'$ is only followed by contractions and weakenings.

     We know that $(\varphi, \varphi', \eta)$ is a proof subsumption. If $\mathrm{es}(\varphi)\eta$ does not contain the principal formula of $x$, then, by definition of proof subsumption, $(\varphi, \varphi_1', \eta)$ is a proof subsumption. By (IH), there exists a $\chi \in \varphi \bowtie \psi$ such that $(\chi, \chi_1', \eta)$ is a proof subsumption. But also the end-sequent of $\chi\eta$ cannot contain the auxiliary formula of $x$ and so $(\chi, \chi', \eta)$ is a proof subsumption as well.

     If the principal formula of $x$ is in $\mathrm{es}(\varphi)\eta$, then, by definition of proof subsumption, $\varphi =$

$$\cfrac{\begin{matrix}(\varphi_1)\\ \Gamma_1 \vdash \Delta_1, A^m\end{matrix}}{\Gamma \vdash \Delta, A^m}\ x$$

     and $(\varphi_1, \varphi_1', \eta)$ is a proof subsumption. By (IH), there exists a proof $\chi_1 \in \varphi_1 \bowtie \psi$ such that $(\chi_1, \chi_1', \eta)$ is a proof subsumption. We assume that $\chi_1$ is maximally contracted (i.e. all possible contractions admitted in a ground proof resolution are carried out). We distinguish the following cases:

     (a) The rule $x$ has (exactly) one auxiliary formula $F$ in $\Gamma_1 \vdash \Delta_1$. If $F \notin \mathrm{es}(\chi_1)$, then $(\chi_1, \chi', \eta)$ is a proof subsumption and, by definition of ground proof resolution, $\chi_1 \in \varphi \bowtie \psi$.

If $F \in \mathrm{es}(\chi_1)$, we define $\chi =$

$$\dfrac{\begin{array}{c}(\chi_1)\\ \Gamma_1, \Pi_1 \vdash \Delta_1, \Lambda_1\end{array}}{\Gamma_1^*, \Pi_1 \vdash \Delta_1^*, \Lambda_1}\ x$$

Then $\chi \in \varphi \bowtie \psi$ and $(\chi, \chi', \eta)$ is a proof subsumption. Note that, as $\chi_1$ is maximally contracted, it can cope with the contractions in $\chi'$.

(b) $x$ is $\rightarrow_r$.
Let $F_1, F_2$ be the auxilary formulas of $x$ in $\varphi'$. If none of $F_1, F_2$ are in $\mathrm{es}(\chi_1)\eta$, then also $(\chi_1, \chi', \eta)$ is a proof subsumption and $\chi_1 \in \varphi \bowtie \psi$.
If both $F_1, F_2$ are in $\mathrm{es}(\chi_1)\eta$, then we define $\chi =$

$$\dfrac{\begin{array}{c}(\chi_1)\\ \Gamma_1, \Pi_1, F_1^0 \vdash \Delta_1, \Lambda_1, F_2^0\end{array}}{\Gamma_1, \Pi_1 \vdash \Delta_1, \Lambda_1, F_1^0 \rightarrow F_2^0}\ \rightarrow_r$$

where $F_1^0\eta = F_1$ and $F_2^0\eta = F_2$.
Then $(\chi, \chi', \eta)$ is a proof subsumption and $\chi \in \varphi \bowtie \psi$.

In case just one of the formulas $F_1, F_2$ is in $\mathrm{es}(\chi_1)$, we adjust $\chi_1$ by producing the other one by weakening (which is admitted in ground proof resolutions) and define $\chi$ by applying $\rightarrow_r$ afterwards. Then, again, $(\chi, \chi', \eta)$ is a proof subsumption and $\chi \in \varphi \bowtie \psi$.

(c) $x$ is $c_l$:
Let $F$ be the contracted formula in $\varphi'$, i.e. $F, F \in \Gamma'$. If none or just one of these $F$'s is in $\mathrm{es}(\chi_1)\eta$ then $(\chi_1, \chi', \eta)$ is a proof subsumption and $\chi_1 \in \varphi \bowtie \psi$.

So we consider the case where both $F$'s are in $\mathrm{es}(\chi_1)\eta$. Let $F_0$ be the contracted formula in $\varphi$ and $F_0\eta = F$. Then we define $\chi =$

$$\dfrac{\begin{array}{c}(\chi_1)\\ F_0, F_0, \Gamma_1, \Pi_1 \vdash \Delta_1, \Lambda_1\end{array}}{F_0, \Gamma_1, \Pi_1 \vdash \Delta_1, \Lambda_1}\ c_l$$

Then $(\chi, \chi', \eta)$ is a proof subsumption and $\chi \in \varphi \bowtie \psi$.

(d) $x$ is $c_r$: analogous to $c_l$.

(e) $x$ is weakening: trivial.

2. The principal formula of $x$ is in $B^p$. Then, as $B$ is an atom, the only possible rules are $c_r$ and $w_r$.

(a) $x = c_r$. Then $\varphi' =$

$$\cfrac{\begin{matrix}(\varphi'_1)\\ \Gamma' \vdash \Delta', B^{p+1}\end{matrix}}{\Gamma' \vdash \Delta', B^p}\ c_r$$

Then $\varphi' \bowtie_l \psi' = \varphi'_1 \bowtie_l \psi'$ (the left proof resolution on $(B^p, B^q)$, the right one on $(B^{p+1}, B^q)$). The result follows directly from (IH).

(b) $x = w_r$. We distinguish two cases: $p = 1$. Then $\varphi' =$

$$\cfrac{\begin{matrix}(\varphi'_1)\\ \Gamma' \vdash \Delta'\end{matrix}}{\Gamma' \vdash \Delta', B}\ w_r$$

Then $\chi'$ is $\varphi'_1$ (possibly) followed by some weakenings, resulting in a proof of $\Gamma', \Pi'' \vdash \Delta', \Lambda''$ ( a subsequent of $\Gamma', \Delta' \vdash \Delta', \Lambda'$). Then either $(\varphi, \varphi'_1, \eta)$ is a proof subsumption, in which case $(\varphi, \chi', \eta)$ is a proof subsumption and $\varphi \in \varphi \bowtie \psi$. If $(\varphi, \varphi'_1, \eta)$ is not a proof subsumption then $\varphi =$

$$\cfrac{\begin{matrix}(\varphi_1)\\ \Gamma \vdash \Delta\end{matrix}}{\Gamma \vdash \Delta, A}\ w_r$$

Then, by definition of ground proof resolution, $\varphi_1 \in \varphi \bowtie \psi$ and $(\varphi_1, \chi', \eta)$ is a proof subsumption. If $p > 1$ the argument is the same as for $c_r$.

- The last inference in $\varphi'$ is a binary rule: We choose the rule $\vee_l$; the arguments for the other binary rules are analogous. Moreover, we only consider left-resolution (the case of right-resolution is symmetric). so let $\varphi' =$

$$\cfrac{\begin{matrix}(\varphi'_1)\\ C, \Gamma'_1 \vdash \Delta, \Delta'_1, B^{p_1}\end{matrix} \quad \begin{matrix}(\varphi'_2)\\ D, \Gamma'_2 \vdash \Delta, \Delta'_2, B^{p_2}\end{matrix}}{C \vee D, \Gamma'_1, \Gamma'_2 \vdash \Delta, \Delta'_1, \Delta'_2, B^p}\ \vee_l$$

where $p_1 + p_2 = p$ and $C \vee D, \Gamma'_1, \Gamma'_2 \vdash \Delta, \Delta'_1, \Delta'_2, B^p = \Gamma' \vdash \Delta', B^p$.

Now, we consider $\chi' \in \varphi' \bowtie_l \psi'$. Then there exist proofs $\pi_1, \pi_2$ such that

$$\pi_1 \in \begin{matrix}(\varphi'_1)\\ C, \Gamma'_1 \vdash \Delta, \Delta'_1, B^{p_1}\end{matrix} \bowtie \begin{matrix}(\psi')\\ B^q, \Pi' \vdash \Lambda'\end{matrix}$$

$$\pi_2 \in \begin{matrix}(\varphi'_2)\\ D, \Gamma'_2 \vdash \Delta, \Delta'_2, B^{p_2}\end{matrix} \bowtie \begin{matrix}(\psi')\\ B^q, \Pi' \vdash \Lambda'\end{matrix}$$

and $\chi' =$

$$\cfrac{\cfrac{\begin{matrix}(\pi_1)\\ C, \Gamma''_1, \Pi'' \vdash \Delta, \Delta''_1, \Lambda''\end{matrix} \quad \begin{matrix}(\pi_2)\\ D, \Gamma''_2, \Pi^* \vdash \Delta, \Delta''_2, \Lambda^*\end{matrix}}{C \vee D, \Gamma'', \Pi'', \Pi^* \vdash \Delta, \Delta''_1, \Delta''_2, \Lambda'', \Lambda^*}\ \vee_l}{S'}\ s^*$$

where $s^*$ is a sequence of weakenings and contractions.

We know that $(\varphi, \varphi', \eta)$ is a proof subsumption. We distinguish several cases:

1. $\mathrm{es}(\varphi)\eta \subseteq \Gamma_1', \Gamma_2' \vdash \Delta, \Delta_1', \Delta_2', B^p$.
   Then, by Definition 5.3, either $(\varphi^*, \varphi_1', \eta)$ or $(\varphi^*, \varphi_2', \eta)$ for some $\varphi^*$ such that $\varphi =$
   $$\frac{\begin{array}{c}\varphi^* \\ S_0\end{array}}{S}\ w^*$$
   is a proof subsumption. We consider the case that $(\varphi^*, \varphi_1', \eta)$ is a proof subsumption, the other case is symmetric.

   Now consider the ground proof resolution $\pi_1$. By (IH), there exists a $\chi \in \varphi^* \bowtie \psi$ such that $(\chi, \pi_1, \eta)$ is a proof subsumption. In particular, we obtain
   $$\mathrm{es}(\chi)\eta \subseteq \Gamma_1'', \Pi'' \vdash \Delta, \Delta_1'', \Lambda''.$$
   and so $(\chi, \chi', \eta)$ is a proof subsumption as well. But we also have $\chi \in \varphi \bowtie \psi$, by definition of ground proof resolution over weakenings.

2. $C \vee D \in \mathrm{es}(\varphi\eta)$.
   Then, by Definition 5.3, the last rule in $\varphi$ must be $\vee_l$. There are several cases to consider:
   (a) $C$ occurs in a premise of $\vee_l$ and $D$ does not,

   (b) $D$ occurs in a premise of $\vee_l$ and $C$ does not,

   (c) both $C$ and $D$ occur in the premises of $\vee_l$.

   In case 2a or 2b the missing auxiliary formula is added by weakening in order to perform $\vee_l$; otherwise the form of $\varphi$ is like in case 2c. Thus, we consider case 2c only. In this case, $\varphi =$
   $$\frac{\begin{array}{c}(\varphi_1) \\ C_0, \Gamma_1 \vdash \Delta_0, \Delta_1, A^{m_1}\end{array} \quad \begin{array}{c}(\varphi_2) \\ D_0, \Gamma_2 \vdash \Delta_0, \Delta_2, A^{m_2}\end{array}}{C_0 \vee C_0, \Gamma_1, \Gamma_2 \vdash \Delta_0, \Delta_1, \Delta_2, A^m}\ \vee_l$$
   where $m_1 + m_2 = m,\ \Delta_0\eta \subseteq \Delta,\ C_0\eta = C,\ D_0\eta = D$ and $(\varphi_1, \varphi_1', \eta)$, $(\varphi_2, \varphi_2', \eta)$ are both proof subsumptions.
   By (IH), there exist ground proof resolutions $\chi_1 \in \varphi_1 \bowtie \psi$ and $\chi_2 \in \varphi_2 \bowtie \psi$ such that
   $$(\chi_1, \pi_1, \eta) \text{ and } (\chi_2, \pi_2, \eta) \text{ are proof subsumptions.}$$
   We define $\chi$ as
   $$\frac{\dfrac{\begin{array}{c}(\chi_1) \\ C_0, \Gamma_1^+, \Pi^+ \vdash \Delta_0, \Delta_1^+, \Lambda^+\end{array} \quad \begin{array}{c}(\chi_2) \\ D_0, \Gamma_2^+, \Pi^\star \vdash \Delta_0, \Delta_2^+, \Lambda^\star\end{array}}{C_0 \vee D_0, \Gamma_1^+, \Gamma_2^+, \Pi^+, \Pi^\star \vdash \Delta_0, \Delta_1^+, \Delta_2^+, \Lambda^+, \Lambda^\star}\ \vee_l}{S}\ c^*$$

where $c^*$ is a maximal sequence of contractions (i.e. no further contractions are applicable to $S$). Then $\chi \in \varphi \bowtie \psi$ and, by Definition 5.3 (binary case), $(\chi, \chi', \eta)$ is a proof subsumption.

□

**Theorem 6.1** (Lifting theorem for proofs)**.** *Let $(\varphi, \varphi', \vartheta_1)$ and $(\psi, \psi', \vartheta_2)$ be proof subsumptions for cut-free proofs which are variable-disjoint. Let $\chi'$ be a ground proof resolution of $\varphi'$ and $\psi'$. Then either*

- *$(\varphi, \chi', \vartheta_1)$ is a proof subsumption or*
- *$(\psi, \chi', \vartheta_2)$ is a proof subsumption or*
- *There exists a resolvent $\chi$ of $\varphi$ and $\psi$ and a substitution $\sigma$ such that $(\chi, \chi', \sigma)$ is a proof subsumption.*

*Proof.* Let $\mathrm{es}(\varphi') = \Gamma \vdash \Delta, A^m$ and $\mathrm{es}(\psi') = A^n, \Pi \vdash \Lambda$ and $\chi'$ be an $A$-resolution of $\varphi'$ and $\psi'$. We distinguish the following cases:

(a) $\mathrm{es}(\varphi)\vartheta_1 \subseteq \Gamma \vdash \Delta$. Then $(\varphi, \chi', \vartheta_1)$ is a proof subsumption by Lemma 6.1.

(b) $\mathrm{es}(\psi)\vartheta_2 \subseteq \Pi \vdash \Lambda$. Then $(\psi, \chi', \vartheta_2)$ is a proof subsumption by Lemma 6.2.

(c) $\mathrm{es}(\varphi)\vartheta_1 \not\subseteq \Gamma \vdash \Delta$ and $\mathrm{es}(\psi)\vartheta_2 \not\subseteq \Pi \vdash \Lambda$. As $\varphi, \psi$ are variable-disjoint, there exists a substitution $\vartheta$ such that $(\varphi, \varphi', \vartheta)$ and $(\psi, \psi', \vartheta)$ are proof subsumptions. In this case, $\varphi$ must be a proof of $\Gamma_0 \vdash \Delta_0, A_1, \ldots, A_k$, $\psi$ a proof of $B_1, \ldots, B_l, \Pi_0 \vdash \Lambda_0$ such that $1 \leq k \leq m, 1 \leq l \leq n$, and

$$\begin{aligned}&\Gamma_0\vartheta \subseteq \Gamma,\ \Delta_0\vartheta \subseteq \Delta,\ A_1\vartheta = \ldots, A_k\vartheta = A,\\ &\Pi_0\vartheta \subseteq \Pi,\ \Lambda_0\vartheta \subseteq \Lambda \text{ and } B_1\vartheta = \ldots B_l\vartheta = A.\end{aligned}$$

Therefore, $\vartheta$ is a unifier of $\{A_1, \ldots, A_k, B_1, \ldots, B_l\}$ and there exists a most general unifier $\lambda$ of this set; hence, there exists a substitution $\sigma$ such that $\lambda\sigma = \vartheta$. Let $\{B\} = \{A_1, \ldots, A_k, B_1, \ldots, B_l\}\lambda$.
Then $\varphi\lambda$ is a proof of $\Gamma_0\lambda \vdash \Delta_0\lambda, B^k$, $\psi\lambda$ is a proof of $B^l, \Pi_0\lambda \vdash \Lambda_0\lambda$ and

$$(\varphi\lambda, \varphi', \sigma),\ (\psi\lambda, \psi', \sigma) \text{ are proof subsumptions.}$$

Now, let $\chi'$ be a ground proof resolution of $\varphi'$ and $\psi'$. Then, by Lemma 6.3, there exists a ground proof resolution $\chi$ of $\varphi\lambda$ and $\psi\lambda$ such that $(\chi, \chi', \sigma)$ is a proof subsumption. By Definition 6.2, $\chi$ is a resolvent of $\varphi$ and $\psi$.

□

This result guarantees that the resolution of the projections of the $\mathrm{ACNF}^{\mathrm{top}}$-normal forms can be "lifted" to the original proof.

## 7. Ground Completeness

In the previous section, we have introduced the notion of ground proof resolution, which combines two cut-free proofs via elimination of certain atoms using proof resolution. Towards our initial goal, namely the completeness of `CERES` for intuitionistic logic, we will first prove—as an intermediary step—the completeness of the proof resolution procedure on the ground level by considering proofs in $\mathrm{ACNF}^{\mathrm{top}}$ enriched by indices. These indices serve the purpose of keeping track of atoms that originate from the same cut in the original **LJ**-proof. Moreover, we will restrict the resolution of two projections on the ground level in such a way that the atoms upon which we perform the resolution must have the same index.

First, we introduce a special kind of $\mathrm{ACNF}^{\mathrm{top}}$ on which the completeness proof relies.

**Definition 7.1 (Chain of Atomic Cuts).** We inductively define a *chain of atomic cuts* as follows:
The base case is given by

$$\frac{A \vdash A^i \quad A^i \vdash A}{A \vdash A}\ cut,$$

where $A$ is an atom and $i \in \mathbb{N}$. We refer to this case as a *trivial chain of atomic cuts.*
Assume that $\sigma_1$ and $\sigma_2$ are chains of atomic cuts with end-sequents $A \vdash A^j$ and $A^j \vdash A$, respectively. Then

$$\frac{\begin{matrix}(\sigma_1)\\ A \vdash A^j\end{matrix} \quad \begin{matrix}(\sigma_2)\\ A^j \vdash A\end{matrix}}{A \vdash A}\ cut$$

is a chain of atomic cuts.

Let $\varphi$ be an $\mathrm{ACNF}^{\mathrm{top}}$, then we say that $\varphi$ is a *proof with nontrivial chains of atomic cuts* if $\varphi$ contains nontrivial chains of atomic cuts at the top.

In order to simplify the completeness proof, we will consider the following special forms of projections obtained from an $\mathrm{ACNF}^{\mathrm{top}}$.

**Definition 7.2 ($\mathrm{ACNF}^{\mathrm{top}}$-projection).** Let $\psi$ be an $\mathrm{ACNF}^{\mathrm{top}}$ of an **LJ**-proof $\varphi$. Then we define, for each $C = \Pi \vdash \Lambda \in \mathrm{CL}(\psi)$, its projection $\psi[C]$ as follows:

(a) For $\psi[\Pi' \vdash \Lambda', D^i]$, replace

$$\frac{D \vdash D^i \quad D^i \vdash D}{D \vdash D}\ cut \qquad \text{in } \psi \text{ by} \qquad \frac{D \vdash D^i}{D \vdash D^i, D}\ w_r.$$

(b) For $\psi[D^i, \Pi' \vdash \Lambda']$, replace

$$\frac{D \vdash D^i \quad D^i \vdash D}{D \vdash D}\ cut \qquad \text{in } \psi \text{ by} \qquad \frac{D^i \vdash D}{D, D^i \vdash D}\ w_l.$$

(c) For $\psi[\Pi' \vdash \Lambda', D^i]$, replace

$$\dfrac{D \vdash D^i \quad D^i \vdash D^j}{D \vdash D^j}\,cut \qquad \text{in } \psi \text{ by} \qquad \dfrac{\dfrac{D \vdash D^i}{D \vdash D^i, D^j}\,w_r}{D \vdash D^i, D^j, D}\,w_r.$$

(d) For $\psi[D^i, \Pi' \vdash \Lambda', D^j]$, replace

$$\dfrac{D \vdash D^i \quad D^i \vdash D^j}{D \vdash D^j}\,cut \qquad \text{in } \psi \text{ by} \qquad \dfrac{\dfrac{D^i \vdash D^j}{D, D^i \vdash D^j}\,w_l}{D, D^i \vdash D^j, D}\,w_r.$$

(e) For $\psi[D^i, \Pi' \vdash \Lambda', D^j]$, replace

$$\dfrac{D^i \vdash D^j \quad D^j \vdash D^l}{D^i \vdash D^l}\,cut \qquad \text{in } \psi \text{ by} \qquad \dfrac{\dfrac{\dfrac{D^i \vdash D^j}{D^i \vdash D^j, D^l}\,w_r}{D^i \vdash D^j, D^l, D}\,w_r}{D, D^i \vdash D^j, D^l, D}\,w_l.$$

(f) For $\psi[D^j, \Pi' \vdash \Lambda', D^l]$, replace

$$\dfrac{D^i \vdash D^j \quad D^j \vdash D^l}{D^i \vdash D^l}\,cut \qquad \text{in } \psi \text{ by} \qquad \dfrac{\dfrac{\dfrac{D^j \vdash D^l}{D^i, D^j \vdash D^l}\,w_l}{D^i, D^j \vdash D^l, D}\,w_r}{D, D^i, D^j \vdash D^l, D}\,w_l.$$

(g) For $\psi[D^i, \Pi' \vdash \Lambda', D^j]$, replace

$$\dfrac{D^i \vdash D^j \quad D^j \vdash D}{D^i \vdash D}\,cut \qquad \text{in } \psi \text{ by} \qquad \dfrac{\dfrac{D^i \vdash D^j}{D^i \vdash D^j, D}\,w_r}{D, D^i \vdash D^j, D}\,w_l.$$

(h) For $\psi[D^j, \Pi' \vdash \Lambda']$, replace

$$\dfrac{D^i \vdash D^j \quad D^j \vdash D}{D^i \vdash D}\,cut \qquad \text{in } \psi \text{ by} \qquad \dfrac{\dfrac{D^j \vdash D}{D^i, D^j \vdash D}\,w_l}{D, D^i, D^j \vdash D}\,w_l.$$

As all inferences below the atomic cuts in $\psi$ do not operate on cut ancestors, we include all of them in the projections. Clearly, $\psi[C]$ is then a proof of $\Gamma, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, \Delta$, where $\Gamma_{\mathsf{ind}}$ and $\Delta_{\mathsf{ind}}$ solely contain indexed atoms.

Note that it is possible for two different atomic cuts within an $\mathrm{ACNF}^{\mathrm{top}}$ to have cut-formulas with the same index.

The following two lemmas will give us a constructive method for resolving two $\mathrm{ACNF}^{\mathrm{top}}$-projections that only differ on the position of a specific indexed atom.

**Lemma 7.1.** *Let $\psi$ be an* $\mathrm{ACNF}^{\mathrm{top}}$ *of an* **LJ**-*proof $\varphi$ (without nontrivial chains of atomic cuts) and let $\psi'$ be a proof obtained from $\psi$ by eliminating an uppermost atomic cut. Then for all $C \in \mathrm{CL}(\psi')$ there are $C_1, C_2 \in \mathrm{CL}(\psi)$ such that $\psi'[C] \in \psi[C_1] \bowtie \psi[C_2]$.*

*Proof.* For a proof in $\mathrm{ACNF}^{\mathrm{top}}$, for each cut-formula $A^i$, there are $C_1, C_2 \in \mathrm{CL}(\psi)$ such that $C_1 = \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, A^i$ and $C_2 = A^i, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}$, where $\Gamma_{\mathsf{ind}}$ and $\Delta_{\mathsf{ind}}$ are possibly empty multisets of indexed atoms.

Thus, let $C := \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}$, i.e. $C \in \mathrm{CL}(\psi')$, by definition of CL and $\mathrm{ACNF}^{\mathrm{top}}$. Then, by definition of $\mathrm{ACNF}^{\mathrm{top}}$ projections, there is a branch going from the end-sequent to the axiom containing the atom $A^i$ such that $\psi[C_1]$ and $\psi[C_2]$ only differ on the position of $A^i$ in this branch. This means, $\psi[C_1]$ and $\psi[C_2]$ are, without loss of generality, of the following forms

$$\begin{array}{c} \dfrac{A \vdash A^i}{A \vdash A^i, A}\,w_r \quad \text{other axioms} \\ \vdots \\ \Gamma, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, \Delta, A^i \end{array} \qquad \text{and} \qquad \begin{array}{c} \dfrac{A^i \vdash A}{A, A^i \vdash A}\,w_l \quad \text{other axioms} \\ \vdots \\ A^i, \Gamma, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, \Delta \end{array}$$

Since these two proofs are projections of the same proof, they only differ in the position of the indexed atom $A^i$. Thus, we can construct an **LJ**-proof $\chi \in \psi[C_1] \bowtie \psi[C_2]$ as follows:

1. Select either $\psi[C_1]$ or $\psi[C_2]$ and go to step 2.
2. Omit both the uppermost axiom together with the uppermost application of weakening in the chosen projection.
3. Omit all occurrences of $A^i$ in the chosen projection and leave the other inferences unchanged.

The result of this procedure is a proof $\chi$ of the following form

$$\begin{array}{c} A \vdash A \quad \text{other axioms} \\ \vdots \\ \Gamma, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, \Delta \end{array}$$

which, by definition of $\mathrm{ACNF}^{\mathrm{top}}$, corresponds to the projection of $\psi'$ to $C = \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}$. Hence, $\chi = \psi'[C] \in \psi[C_1] \bowtie \psi[C_2]$. ☐

**Lemma 7.2.** *Let $\psi$ be an* $\mathrm{ACNF}^{\mathrm{top}}$ *of an* **LJ**-*proof $\varphi$, and let $\psi'$ be a proof obtained from $\psi$ by eliminating an uppermost atomic cut. Then for all $C \in \mathrm{CL}[\psi']$ there are $C_1, C_2 \in \mathrm{CL}(\psi)$ such that $\psi'[C] \in \psi[C_1] \bowtie \psi[C_2]$.*

*Proof.* If $\psi$ does not contain nontrivial chains of atomic cuts, then the result follows from Lemma 7.1. Thus, for the remaining part of the proof, we assume that $\psi$ contains nontrivial chains of atomic cuts. We distinguish cases according to the structure of the uppermost cut:

1. (Definition 7.2, cases c & d)

$$\dfrac{D \vdash D^i \quad D^i \vdash D^j}{D \vdash D^j}\,cut$$

In this case, there are $C_1, C_2 \in \mathrm{CL}(\psi)$ such that $C_1 = \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, D^i$ and $C_2 = D^i, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, D^j$, where $\Gamma_{\mathsf{ind}}, \Delta_{\mathsf{ind}}$ are possibly empty multisets of indexed atoms. Thus, let $C := \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, D^j$, and observe that $C \in \mathrm{CL}(\psi')$, by definition of CL and $\mathrm{ACNF}^{\mathrm{top}}$. Consequently, by definition of $\mathrm{ACNF}^{\mathrm{top}}$ projections, the corresponding projections $\psi[C_1]$ and $\psi[C_2]$ are, without loss of generality, of the following forms

$$\begin{array}{ccc} \dfrac{\dfrac{D \vdash D^i}{D \vdash D^i, D^j}\, w_r}{D \vdash D^i, D^j, D}\, w_r & \text{other axioms} \\ \vdots & \\ \Gamma, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, \Delta, D^i, D^j & \end{array} \qquad \begin{array}{cc} \dfrac{\dfrac{D^i \vdash D^j}{D, D^i \vdash D^j}\, w_l}{D, D^i \vdash D^j, D}\, w_r & \text{other axioms} \\ \vdots & \\ D^i, \Gamma, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, \Delta, D^j & \end{array}$$

Since both of these proofs are projections of the same proof, they only differ on a branch that starts in the end-sequent and ends in the axioms $D \vdash D^i$ and $D^i \vdash D^j$, respectively. The respective branches in these two projections only differ in the position of the indexed atom $D^i$ and on the uppermost inference.
Hence, we can construct an **LJ**-proof $\chi \in \psi[C_1] \bowtie \psi[C_2]$ as follows:

(a) Select either $\psi[C_1]$ or $\psi[C_2]$ and go to step 2.
(b) Omit both the uppermost axiom together with the uppermost application of weakening in the chosen projection.
(c) Omit all occurrences of $D^i$ in the chosen projection and leave the other inferences unchanged.

The result of this procedure is a proof $\chi$ of the following form

$$\begin{array}{cc} \dfrac{D \vdash D^j}{D \vdash D^j, D}\, w_r & \text{other axioms} \\ \vdots & \\ \Gamma, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, \Delta, D^j & \end{array}$$

which, by definition of $\mathrm{ACNF}^{\mathrm{top}}$, corresponds to the projection of $\psi'$ to $C = \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, D^j$.
Hence, $\chi = \psi'[C] \in \psi[C_1] \bowtie \psi[C_2]$.

2. (Definition 7.2, cases e & f)

$$\dfrac{D^i \vdash D^j \quad D^j \vdash D^l}{D^i \vdash D^l}\, cut$$

Analogous to Case 1, we have $C_1, C_2 \in \mathrm{CL}(\psi)$ such that $C_1 = D^i, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, D^j$ and $C_2 = D^j, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, D^l$ as well as $C := D^i, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, D^l$. The corresponding projections $\psi[C_1]$ and $\psi[C_2]$ are of the following forms

$$\begin{array}{cc} \dfrac{\dfrac{\dfrac{D^i \vdash D^j}{D^i \vdash D^j, D^l}\, w_r}{D^i \vdash D^j, D^l, D}\, w_r}{D, D^i \vdash D^j, D^l, D}\, w_l & \text{other axioms} \\ \vdots & \\ D^i, \Gamma, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, \Delta, D^j, D^l & \end{array} \qquad \begin{array}{cc} \dfrac{\dfrac{\dfrac{D^j \vdash D^l}{D^i, D^j \vdash D^l}\, w_l}{D^i, D^j \vdash D^l, D}\, w_r}{D, D^i, D^j \vdash D^l, D}\, w_l & \text{other axioms} \\ \vdots & \\ D^i, D^j, \Gamma, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, \Delta, D^l & \end{array}$$

Applying steps (a)–(c) of the above procedure yields, in this case, a proof $\chi$ of the following form

$$\dfrac{\dfrac{\dfrac{D^i \vdash D^l}{D^i \vdash D^l, D}\, w_r}{D, D^i \vdash D^l, D}\, w_l \quad \text{other axioms}}{\vdots \atop D^i, \Gamma, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, \Delta, D^l}$$

which, by definition of ACNF$^{\text{top}}$, corresponds to the projection of $\psi'$ to $C = D^i, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, D^l$.
Hence, $\chi = \psi'[C] \in \psi[C_1] \bowtie \psi[C_2]$.

3. (Definition 7.2, cases g & h)

$$\frac{D^i \vdash D^j \quad D^j \vdash D}{D^i \vdash D}\, cut$$

Analogous to Cases 1 and 2, we have $C_1, C_2 \in \mathrm{CL}(\psi)$ such that $C_1 = D^i, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, D^j$ and $C_2 = D^j, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}$ as well as $C := D^i, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}$. The projections $\psi[C_1]$ and $\psi[C_2]$ are of the following forms

$$\dfrac{\dfrac{\dfrac{D^i \vdash D^j}{D^i \vdash D^j, D}\, w_r}{D, D^i \vdash D^j, D}\, w_l \quad \text{other axioms}}{\vdots \atop D^i, \Gamma, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, \Delta, D^j} \qquad \dfrac{\dfrac{\dfrac{D^j \vdash D}{D^i, D^j \vdash D}\, w_l}{D, D^i, D^j \vdash D}\, w_l \quad \text{other axioms}}{\vdots \atop D^i, D^j, \Gamma, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, \Delta}$$

Applying steps (a)–(c) of the above procedure yields, in this case, a proof $\chi$ of the following form

$$\dfrac{\dfrac{D^i \vdash D}{D, D^i \vdash D}\, w_l \quad \text{other axioms}}{\vdots \atop D^i, \Gamma, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}, \Delta}$$

which, by definition of ACNF$^{\text{top}}$, corresponds to the projection of $\psi'$ to $C = D^i, \Gamma_{\mathsf{ind}} \vdash \Delta_{\mathsf{ind}}$.
Hence, $\chi = \psi'[C] \in \psi[C_1] \bowtie \psi[C_2]$.

$\square$

**Definition 7.3.** Let $\psi$ be an ACNF$^{\text{top}}$ of an **LJ**-proof $\varphi$ and let $\psi'$ be obtained from $\psi$ by eliminating a single atomic cut with cut-formula $A^i$. Then we write $\mathcal{P}(\psi) \Rightarrow^{\bowtie} \mathcal{P}(\psi')$ if for each $C' \in \mathrm{CL}(\psi')$ we have that there are $C_1, C_2 \in \mathrm{CL}(\psi)$ with $\psi'[C'] \in \psi[C_1] \bowtie \psi[C_2]$ such that $\psi[C_1]$ and $\psi[C_2]$ were resolved upon $A^i$.

The following theorem states that the projections of the clause set after every cut-elimination step performed on a proof in ACNF$^{\text{top}}$ can be obtained by simply resolving the previous projections using the methods established in Lemmas 7.1 and 7.2.

**Theorem 7.1.** *Let $\psi$ be an* $\mathrm{ACNF}^{\mathrm{top}}$ *of an* **LJ***-proof $\varphi$. Then proof resolution yields the projections of each* $\mathrm{CL}(\psi^i)$*, where $\psi^{i+1}$ is obtained from $\psi^i$ by eliminating an uppermost atomic cut, i.e.* $\mathcal{P}(\psi) \Rightarrow^{\bowtie} \mathcal{P}(\psi^1) \Rightarrow^{\bowtie} \ldots \Rightarrow^{\bowtie} \{\psi^*[\vdash]\}$.

*Proof.* We proceed by induction on the number $n$ of atomic cuts in $\psi$.

**Base case:** $n = 0$. In this case, $\psi$ is an **LJ**-proof and $\mathrm{CL}(\psi) = \{\vdash\}$. The only projection of $\psi$ is $\psi[\vdash]$ where $\psi[\vdash] = \psi$.

**Induction Hypothesis (IH):** Assume the claim holds for all $\mathrm{ACNF}^{\mathrm{top}}$'s $\psi$ of $\varphi$ containing $k$ atomic cuts, for $k \leq n$.

**Induction Step:** Let $\psi$ be an $\mathrm{ACNF}^{\mathrm{top}}$ (of an **LJ**-proof $\varphi$) containing $n+1$ atomic cuts. Furthermore, let $\psi^1$ be a proof obtained by reductively eliminating an uppermost atomic cut with cut-formula $A^{i_j}$, for $i_j \in \{i_1, \ldots, i_{n+1}\}$, from $\psi$. Now, let $\psi^1[C]$ be an arbitrary projection in $\mathcal{P}(\psi^1)$. Thus, by Lemma 7.2, there are $\psi[C_1]$, $\psi[C_2] \in \mathcal{P}(\psi)$ such that $\psi^1[C] \in \psi[C_1] \bowtie \psi[C_2]$. Therefore,

$$\mathcal{P}(\psi) \Rightarrow^{\bowtie} \mathcal{P}(\psi^1).$$

Moreover, since $\psi^1$ contains less than $n+1$ atomic cuts, we can apply the (IH) to obtain that $\mathcal{P}(\psi^1) \Rightarrow^{\bowtie} \mathcal{P}(\psi^2) \Rightarrow^{\bowtie} \ldots \Rightarrow^{\bowtie} \{\psi^*[\vdash]\}$. Putting things together, we obtain $\mathcal{P}(\psi) \Rightarrow^{\bowtie} \mathcal{P}(\psi^1) \Rightarrow^{\bowtie} \ldots \Rightarrow^{\bowtie} \{\psi^*[\vdash]\}$. □

Now, we are able to prove the main result of this section, i.e. that proof-resolving all projections arising from the characteristic clause set of an **LJ**-proof in $\mathrm{ACNF}^{\mathrm{top}}$ eventually yields a cut-free **LJ**-proof of the same end-sequent.

**Theorem 7.2.** *Let $\psi$ be an* $\mathrm{ACNF}^{\mathrm{top}}$ *of an* **LJ***-proof $\varphi$ of $\Gamma \vdash \Delta$. Then resolving the projections in $\mathcal{P}(\psi)$—guided by the corresponding resolution refutations—yields a cut-free* **LJ***-proof of $\Gamma \vdash \Delta$.*

*Proof.* Immediate consequence of Theorem 7.1 together with the fact that the only possible projection of $\{\vdash\}$ does not contain any indexed atoms. This directly implies that $\psi^*[\vdash]$ is a cut-free **LJ**-proof of $\Gamma \vdash \Delta$. □

## 8. The method `CERES`-i

Given a skolemized **LJ**-proof $\varphi$ (i.e. an intuitionistic proof without strong quantifier inferences), we proceed as follows:

(1) Compute the set of all proof projections $\mathcal{P}(\varphi)$.

(2) Apply proof resolution until a cut-free intuitionistic proof is reached.

Steps (1) and (2) characterize the roughest form of `CERES`-i, without any refinement to reduce proof search. We show first that `CERES`-i is complete and discuss further refinements afterwards.

**Theorem 8.1** (CERES-i completeness)**.** *Let $\varphi$ be an* **LJ**-*proof of a skolemized end-sequent $S$. Then the application of* CERES-*i to $\varphi$ yields a cut-free* **LJ**-*proof $\chi$ of $S$.*

*Proof.* Consider an ACNF$^{\text{top}}$ (Definition 4.4) $\varphi^*$ of $\varphi$. By definition, $\varphi^*$ is an **LJ**-proof of $S$. By Theorem 7.2, there exists a proof resolution of the projections $\mathcal{P}(\varphi^*)$ yielding a cut-free **LJ**-proof $\psi$. By Theorem 5.2, all projections in $\mathcal{P}(\varphi^*)$ are subsumed by the projections in $\mathcal{P}(\varphi)$. By Lemma 6.1, there exists a proof resolution of $\mathcal{P}(\varphi)$ resulting in a proof $\chi$ that subsumes $\psi$, the proof resolution of $\mathcal{P}(\varphi^*)$. Since we know that $\psi$ is an **LJ**-proof, then, by Theorem 5.1, so is $\chi$. $\square$

**Example 8.1.** Consider the proof $\varphi =$

$$
\begin{prooftree}
\AxiomC{$Px^1 \vdash Px$}
\RightLabel{$\neg_l$}
\UnaryInfC{$Px^1, \neg Px \vdash$}
\RightLabel{$w_r$}
\UnaryInfC{$Px^1, \neg Px \vdash Qx$}
\AxiomC{$Qx \vdash Qx^2$}
\RightLabel{$w_l$}
\UnaryInfC{$Px, Qx \vdash Qx^2$}
\RightLabel{$\vee_l$}
\BinaryInfC{$Px^1, \neg Px \vee Qx \vdash Qx^2$}
\RightLabel{$\forall_l$}
\UnaryInfC{$Px^1, \forall y.(\neg Py \vee Qy) \vdash Qx^2$}
\RightLabel{$\to_r$}
\UnaryInfC{$\forall y.(\neg Py \vee Qy) \vdash (Px^1 \to Qx^2)$}
\RightLabel{$\forall_r$}
\UnaryInfC{$\forall y.(\neg Py \vee Qy) \vdash \forall y.(Py^1 \to Qy^2)$}
\AxiomC{$Pa \vdash Pa^1$}
\AxiomC{$Qa^2 \vdash Qa$}
\RightLabel{$\neg_l$}
\UnaryInfC{$Qa^2, \neg Qa \vdash$}
\RightLabel{$\to_l$}
\BinaryInfC{$Pa, \neg Qa, (Pa^1 \to Qa^2) \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg Qa, (Pa^1 \to Qa^2) \vdash \neg Pa$}
\RightLabel{$\to_r$}
\UnaryInfC{$(Pa^1 \to Qa^2) \vdash \neg Qa \to \neg Pa$}
\RightLabel{$\forall_l$}
\UnaryInfC{$\forall y.(Py^1 \to Qy^2) \vdash \neg Qa \to \neg Pa$}
\RightLabel{$cut$}
\BinaryInfC{$\forall y.(\neg Py \vee Qy) \vdash \neg Qa \to \neg Pa$}
\end{prooftree}
$$

Consider the indexed characteristic clause set below:

$$\mathrm{CL}(\varphi) = \{Px^1 \vdash Qx^2;\ \vdash Pa^1;\ Qa^2 \vdash\}.$$

We have the following indexed resolution refutation:

$$
\begin{prooftree}
\AxiomC{$\vdash Pa^1$}
\AxiomC{$Px^1 \vdash Qx^2$}
\BinaryInfC{$\vdash Qa^2$}
\AxiomC{$Qa^2 \vdash$}
\BinaryInfC{$\vdash$}
\end{prooftree}
$$

and the following minimal projections:

$\varphi[\vdash Pa^1]$

$$
\begin{prooftree}
\AxiomC{$Pa \vdash Pa^1$}
\RightLabel{$w_l$}
\UnaryInfC{$\neg Qa, Pa \vdash Pa^1$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg Qa \vdash Pa^1, \neg Pa$}
\RightLabel{$\to_r$}
\UnaryInfC{$\vdash Pa^1, \neg Qa \to \neg Pa$}
\end{prooftree}
$$

$\varphi[Px^1 \vdash Qx^2]$

$$
\begin{prooftree}
\AxiomC{$Px^1 \vdash Px$}
\RightLabel{$\neg_l$}
\UnaryInfC{$Px^1, \neg Px \vdash$}
\AxiomC{$Qx \vdash Qx^2$}
\RightLabel{$\vee_l$}
\BinaryInfC{$Px^1, \neg Px \vee Qx \vdash Qx^2$}
\RightLabel{$\forall_l$}
\UnaryInfC{$Px^1, \forall y(\neg Py \vee Qy) \vdash Qx^2$}
\end{prooftree}
$$

$\varphi[Qa^2 \vdash]$

$$
\begin{prooftree}
\AxiomC{$Qa^2 \vdash Qa$}
\RightLabel{$\neg_l$}
\UnaryInfC{$Qa^2, \neg Qa \vdash$}
\RightLabel{$w_l$}
\UnaryInfC{$Pa, Qa^2, \neg Qa \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$Qa^2, \neg Qa \vdash \neg Pa$}
\RightLabel{$\to_r$}
\UnaryInfC{$Qa^2 \vdash \neg Qa \to \neg Pa$}
\end{prooftree}
$$

Now, we consider an $\mathrm{ACNF}^{\mathrm{top}}$ $\psi$ of the proof $\varphi$:

$$
\begin{prooftree}
\AxiomC{$Pa \vdash Pa^1$}
\AxiomC{$Pa^1 \vdash Pa$}
\RightLabel{$cut$}
\BinaryInfC{$Pa \vdash Pa$}
\RightLabel{$\neg_l$}
\UnaryInfC{$Pa, \neg Pa \vdash$}
\RightLabel{$w_r$}
\UnaryInfC{$Pa, \neg Pa \vdash Qa$}
\AxiomC{$Qa \vdash Qa^2$}
\AxiomC{$Qa^2 \vdash Qa$}
\RightLabel{$cut$}
\BinaryInfC{$Qa \vdash Qa$}
\RightLabel{$\vee_l$}
\BinaryInfC{$Pa, \neg Pa \vee Qa \vdash Qa$}
\RightLabel{$\forall_l$}
\UnaryInfC{$Pa, \forall y(\neg Py \vee Qy) \vdash Qa$}
\RightLabel{$\neg_l$}
\UnaryInfC{$Pa, \neg Qa, \forall y(\neg Py \vee Qy) \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\forall y(\neg Py \vee Qy), \neg Qa \vdash \neg Pa$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$\forall y(\neg Py \vee Qy) \vdash \neg Qa \rightarrow \neg Pa$}
\end{prooftree}
$$

The characteristic clause set of $\psi$ is

$$\mathrm{CL}(\psi) = \{Pa^1 \vdash Qa^2;\; Pa^1, Qa^2 \vdash;\; \vdash Pa^1, Qa^2;\; Qa^2 \vdash Pa_1\}$$

This clause set is typical for an $\mathrm{ACNF}^{\mathrm{top}}$. Clearly $\mathrm{CL}(\varphi) \leq_{ss} \mathrm{CL}(\psi)$.

Now, consider the (very redundant) projections of $\psi$ with respect to $\mathrm{CL}(\psi)$:

$\psi[\vdash Pa^1, Qa^2]$

$$
\begin{prooftree}
\AxiomC{$Pa \vdash Pa^1$}
\RightLabel{$w_r$}
\UnaryInfC{$Pa \vdash Pa^1, Pa$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg Pa, Pa \vdash Pa^1$}
\RightLabel{$w_r$}
\UnaryInfC{$\neg Pa, Pa \vdash Pa^1, Qa$}
\AxiomC{$Qa \vdash Qa^2$}
\RightLabel{$w_r$}
\UnaryInfC{$Qa \vdash Qa^2, Qa$}
\RightLabel{$\vee_l$}
\BinaryInfC{$Pa, \neg Pa \vee Qa \vdash Pa^1, Qa^2, Qa$}
\RightLabel{$\forall_l$}
\UnaryInfC{$Pa, \forall y(\neg Py \vee Qy) \vdash Pa^1, Qa^2, Qa$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg Qa, Pa, \forall y(\neg Py \vee Qy) \vdash Pa^1, Qa^2$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg Qa, \forall y(\neg Py \vee Qy) \vdash Pa^1, Qa^2, \neg Pa$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$\forall y(\neg Py \vee Qy) \vdash Pa^1, Qa^2, \neg Qa \rightarrow \neg Pa$}
\end{prooftree}
$$

$\psi[Pa^1 \vdash Qa^2]$

$$
\begin{prooftree}
\AxiomC{$Pa^1 \vdash Pa$}
\RightLabel{$w_l$}
\UnaryInfC{$Pa, Pa^1 \vdash Pa$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg Pa, Pa, Pa^1 \vdash$}
\RightLabel{$w_r$}
\UnaryInfC{$\neg Pa, Pa, Pa^1 \vdash Qa$}
\AxiomC{$Qa \vdash Qa^2$}
\RightLabel{$w_r$}
\UnaryInfC{$Qa \vdash Qa^2, Qa$}
\RightLabel{$\vee_l$}
\BinaryInfC{$Pa, Pa^1, \neg Pa \vee Qa \vdash Qa^2, Qa$}
\RightLabel{$\forall_l$}
\UnaryInfC{$Pa, Pa^1, \forall y(\neg Py \vee Qy) \vdash Qa^2, Qa$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg Qa, Pa, Pa^1, \forall y(\neg Py \vee Qy) \vdash Qa^2$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg Qa, Pa^1, \forall y(\neg Py \vee Qy) \vdash Qa^2, \neg Pa$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$Pa^1, \forall y(\neg Py \vee Qy) \vdash Qa^2, \neg Qa \rightarrow \neg Pa$}
\end{prooftree}
$$

$\psi[Pa^1, Qa^2 \vdash]$

$$
\begin{prooftree}
\AxiomC{$Pa^1 \vdash Pa$}
\RightLabel{$w_l$}
\UnaryInfC{$Pa, Pa^1 \vdash Pa$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg Pa, Pa, Pa^1 \vdash$}
\RightLabel{$w_r$}
\UnaryInfC{$\neg Pa, Pa, Pa^1 \vdash Qa$}
\AxiomC{$Qa^2 \vdash Qa$}
\RightLabel{$w_l$}
\UnaryInfC{$Qa, Qa^2 \vdash Qa$}
\RightLabel{$\vee_l$}
\BinaryInfC{$Pa, Pa^1, Qa^2, \neg Pa \vee Qa \vdash Qa$}
\RightLabel{$\forall_l$}
\UnaryInfC{$Pa, Pa^1, Qa^2, \forall y(\neg Py \vee Qy) \vdash Qa$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg Qa, Pa, Pa^1, Qa^2, \forall y(\neg Py \vee Qy) \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg Qa, Pa^1, Qa^2, \forall y(\neg Py \vee Qy) \vdash \neg Pa$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$Pa^1, Qa^2, \forall y(\neg Py \vee Qy) \vdash \neg Qa \rightarrow \neg Pa$}
\end{prooftree}
$$

$\psi[Qa^2 \vdash Pa^1]$

$$
\begin{prooftree}
\AxiomC{$Pa \vdash Pa^1$}
\RightLabel{$w_r$}
\UnaryInfC{$Pa \vdash Pa^1, Pa$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg Pa, Pa \vdash Pa^1$}
\RightLabel{$w_r$}
\UnaryInfC{$\neg Pa, Pa \vdash Pa^1, Qa$}
\AxiomC{$Qa^2 \vdash Qa$}
\RightLabel{$w_l$}
\UnaryInfC{$Qa, Qa^2 \vdash Qa$}
\RightLabel{$\vee_l$}
\BinaryInfC{$Pa, Qa^2, \neg Pa \vee Qa \vdash Pa^1, Qa$}
\RightLabel{$\forall_l$}
\UnaryInfC{$Pa, Qa^2, \forall y(\neg Py \vee Qy) \vdash Pa^1, Qa$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg Qa, Pa, Qa^2, \forall y(\neg Py \vee Qy) \vdash Pa^1$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg Qa, Qa^2, \forall y(\neg Py \vee Qy) \vdash Pa^1, \neg Pa$}
\RightLabel{$\rightarrow_r$}
\UnaryInfC{$Qa^2, \forall y(\neg Py \vee Qy) \vdash Pa^1, \neg Qa \rightarrow \neg Pa$}
\end{prooftree}
$$

To simulate the elimination of the cut with the atom $Pa^1$ we have to resolve

- $\psi[\vdash Pa^1, Qa^2]$ and $\psi[Pa^1 \vdash Qa^2]$,
- $\psi[Pa^1, Qa^2 \vdash]$ and $\psi[Qa^2 \vdash Pa^1]$.

We first resolve $\psi[\vdash Pa^1, Qa^2]$ and $\psi[Pa^1 \vdash Qa^2]$ by *using the inference order in the proof* $\psi$, resulting in $\psi_1'$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Pa \vdash Pa}{\neg Pa, Pa \vdash}\;\neg_l}{\neg Pa, Pa \vdash Qa}\;w_r \qquad \dfrac{Qa \vdash Qa^2}{Qa \vdash Qa^2, Qa}\;w_r}{Pa, \neg Pa \vee Qa \vdash Qa^2, Qa}\;\vee_l}{Pa, \forall y(\neg Py \vee Qy) \vdash Qa^2, Qa}\;\forall_l}{\neg Qa, Pa, \forall y(\neg Py \vee Qy) \vdash Qa^2}\;\neg_l}{\neg Qa, \forall y(\neg Py \vee Qy) \vdash Qa^2, \neg Pa}\;\neg_r}{\forall y(\neg Py \vee Qy) \vdash Qa^2, \neg Qa \to \neg Pa}\;\to_r$$

Now, $\psi_1'$ is exactly the projection of the proof $\psi'$ (obtained by eliminating the cut with $Pa^1$) to $\vdash Qa^2$, where $\mathrm{CL}(\psi') = \{\vdash Qa^2;\ Qa^2 \vdash\}$. Indeed, $\psi'$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Pa \vdash Pa}{Pa, \neg Pa \vdash}\;\neg_l}{Pa, \neg Pa \vdash Qa}\;w_r \qquad \dfrac{Qa \vdash Qa^2 \quad Qa^2 \vdash Qa}{Qa \vdash Qa}\;cut}{Pa, \neg Pa \vee Qa \vdash Qa}\;\vee_l}{Pa, \forall y(\neg Py \vee Qy) \vdash Qa}\;\forall_l}{Pa, \forall y(\neg Py \vee Qy) \vee Qa \vdash}\;\neg_l}{\forall y(\neg Py \vee Qy), \neg Qa \vdash \neg Pa}\;\neg_r}{\forall y(\neg Py \vee Qy) \vdash \neg Qa \to \neg Pa}\;\to_r$$

In the same way, we build the proof resolution of $\psi[Pa^1, Qa^2 \vdash]$ and $\psi[Qa^2 \vdash Pa^1]$ (again respecting the inference order in $\psi$) and obtain $\psi_2'$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Pa \vdash Pa}{\neg Pa, Pa \vdash}\;\neg_l}{\neg Pa, Pa \vdash Qa}\;w_r \qquad \dfrac{Qa^2 \vdash Qa}{Qa, Qa^2 \vdash Qa}\;w_l}{Pa, Qa^2, \neg Pa \vee Qa \vdash Qa}\;\vee_l}{Pa, Qa^2, \forall y(\neg Py \vee Qy) \vdash Qa}\;\forall_l}{\neg Qa, Pa, Qa^2, \forall y(\neg Py \vee Qy) \vdash}\;\neg_l}{\neg Qa, Qa^2, \forall y(\neg Py \vee Qy) \vdash \neg Pa}\;\neg_r}{Qa^2, \forall y(\neg Py \vee Qy) \vdash \neg Qa \to \neg Pa}\;\to_r$$

which is, of course, the projection of $\psi'$ to the clause $Qa^2 \vdash$. As $\psi'$ has one cut less than $\psi$ we may apply the induction hypothesis and obtain an intuitionistic proof by resolving the projections $\psi_1'$ and $\psi_2'$ to an intuitionistic proof. This proof is

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Pa \vdash Pa}{Pa, \neg Pa \vdash}\neg_l}{Pa, \neg Pa \vdash Qa}w_r \qquad Qa \vdash Qa}{Pa, \neg Pa \vee Qa \vdash Qa}\vee_l}{Pa, \forall y(\neg Py \vee Qy) \vdash Qa}\forall_l}{Pa, \neg Qa, \forall y(\neg Py \vee Qy) \vdash}\neg_l}{\forall y(\neg Py \vee Qy), \neg Qa \vdash \neg Pa}\neg_r}{\forall y(\neg Py \vee Qy) \vdash \neg Qa \to \neg Pa}\to_r
$$

*which is identical to a proof obtained by reductive elimination.*

Now, we simulate the resolutions on the proof $\varphi$:

- The resolution of $\psi[\vdash Pa^1, Qa^2]$ and $\psi[Pa^1 \vdash Qa^2]$ corresponds to a resolution $\varphi'$ of $\varphi[\vdash Pa^1]$ and $\varphi[Px^1 \vdash Qx^2]$.
- The resolution of $\psi[Pa^1, Qa^2 \vdash]$ and $\psi[Qa^2 \vdash Pa^1]$ corresponds to the projection $\varphi[Qa^2 \vdash]$ itself.

Now, the projection $\psi[\vdash Pa^1, Qa^2]$ is subsumed by $\varphi[\vdash Pa^1]$ and $\psi[Pa^1 \vdash Qa^2]$ by $\varphi[Px^1 \vdash Qx^2]$. This holds also for the proof resolutions which are both of the form $\varphi' = \psi_1'$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Pa \vdash Pa}{Pa, \neg Pa \vdash}\neg_l \qquad \dfrac{Qa \vdash Qa^2}{Qa \vdash Qa^2, Qa}w_l}{Pa, \neg Pa \vee Qa \vdash Qa^2, Qa}\vee_l}{Pa, \forall y(\neg Py \vee Qy) \vdash Qa^2, Qa}\forall_l}{Pa, \forall y(\neg Py \vee Qy), \neg Qa \vdash Qa^2}\neg_l}{\forall y(\neg Py \vee Qy), \neg Qa \vdash \neg Pa, Qa^2}\neg_r}{\forall y(\neg Py \vee Qy) \vdash \neg Qa \to \neg Pa, Qa^2}\to_r
$$

Clearly $\varphi'$ and $\psi[Qa^2 \vdash]$ then resolve to the same cut-free proof as obtained via the ACNF$^{\text{top}}$ $\psi$.

### *8.1. Epsilonization*

It is a well-known fact that skolemization (or, better, de-skolemization) can generate unsound proofs in intuitionistic logic. We could, in principle, skolemize an **LJ** proof, remove the cuts via `CERES`-i, but we could not de-skolemize the final cut-free proof without running the risk of generating an invalid proof. This problem can be resolved by using the epsilonization method for **LJ** proposed in [17]. In this paper, the authors define **LJ**$^\star$, a modified version of **LJ** where the terms used for weak quantifier rules ($\forall_l$ and $\exists_r$) must satisfy the so called *accessibility condition.* Intuitively, this condition encodes the fact that any eigenvariable used in the term was already introduced by a strong quantifier rule below. This allows for the de-epsilonization process to introduce the strong quantifiers at the right places. We describe briefly how the epsilonization procedure can be integrated in `CERES`-i.

Transforming an **LJ** proof into an **LJ**$^\star$ proof is trivial. Since there are no $\varepsilon$-terms in **LJ**, no accessibility conditions are violated. Then epsilonization can be performed and we will have an **LJ**$^\star$ proof without strong quantifiers. At this point, all operations of `CERES`-i can be performed, treating $\varepsilon$-terms as eigenvariables. The difference between them and actual eigenvariables is simply the information available for de-epsilonization. In fact, this information can even be used to refine the proof resolution process, avoiding rule orderings that would clearly fail when removing $\varepsilon$-terms. Once the cut-free proof is obtained, the proof can be de-epsilonized and transformed back into **LJ** (also not a hard procedure, since the rules are the same and there are no eigenvariable violations, guaranteed by the accessibility conditions).

*8.2. A Refinement of* `CERES`-*i*

Though Theorem 8.1 guarantees that, by iterated resolution of proof projections and their resolvents, we always reach an intuitionistic cut-free proof, the method `CERES`-i is not very efficient. In fact, once we derive a wrong resolvent $\psi$, all further resolvents having $\psi$ as ancestor could be useless and heavy backtracking would be necessary. Below, we define a refinement of proof resolution which is still complete but strongly reduces backtracking.

Consider a `CERES`-projection $\pi$ of a proof $\varphi$ of a sequent $\Gamma \vdash \Delta$ (for $\Delta$ empty or consisting of a formula $A$). Then the end-sequent of $\pi$ is of the form $\mathcal{A}, \Gamma' \vdash \Delta', \mathcal{B}$, where $\Gamma' \vdash \Delta'$ is a subsequent of $\Gamma \vdash \Delta$ and $\mathcal{A} \vdash \mathcal{B}$ is an atomic sequent consisting of atoms from the axioms of $\varphi$. We call $\mathcal{A} \vdash \mathcal{B}$ the *clause part* of $\mathrm{es}(\pi)$. Moreover, every sequent occurring in the proof $\pi$ has a clause part and an *end-sequent part* (the formula occurrences which are ancestors of $\Gamma' \vdash \Delta'$). To clearly distinguish the clause part from the end-sequent part, we may give the atoms of the clause part indices as we have done it in our examples (and leave the formulas which are end-sequent ancestors without indices). Then it is easy to see that the property of partitioning into clause part and end-sequent part is invariant under resolution. Now, let us consider the proof of ground completeness in Section 7: in resolving the projections of an $\mathrm{ACNF}^{\mathrm{top}}$ of $\varphi$ we never produce a resolvent having two or more occurrences of an end-sequent ancestor at the right-hand side of a sequent; indeed if we look at the end-sequent part $S'$ of a sequent $S$ occurring in a resolvent $\psi$ then $S'$ is an intuitionistic sequent. This observation motivates the following definitions:

**Definition 8.1 (`CERES`-proof).** A cut-free proof $\psi$ is a `CERES`-*proof* if it is cut-free and $\mathrm{es}(\psi)$ is of the form $\mathcal{A}, \Gamma \vdash \Delta, \mathcal{B}$, where $\Gamma \vdash \Delta$ is a closed sequent and $\mathcal{A} \vdash \mathcal{B}$ is an atomic sequent consisting of atoms occurring in the axioms of $\psi$, and there are no inferences on ancestors of $\mathcal{A} \vdash \mathcal{B}$ in $\psi$. Let $S$ be a sequent occurring in $\psi$ and $\mathcal{A}' \vdash \mathcal{B}'$ the subsequent of $S$ consisting of ancestors of $\mathcal{A} \vdash \mathcal{B}$; then $\mathcal{A}' \vdash \mathcal{B}'$ is called the *clause part* of $S$ (notation $c(S)$). The remaining part of $S$ is called the *end-sequent part* of $S$ (notation $e(S)$).

The proofs obtained by proof resolution in Section 7 are, in general, not intuitionistic, but all sequents occurring in these proofs have only intuitionistic end-sequent parts. So only the clause parts of the proofs make them non-intuitionistic.

**Definition 8.2 (weakly intuitionistic).** A CERES-proof $\psi$ is called *weakly intuitionistic* if, for all sequents $S$ occurring in $\psi$, $e(S)$ is intuitionistic.

**Proposition 8.1.** *Let $\varphi, \psi$ be* CERES*-proofs and $\chi$ be a resolvent of $\varphi$ and $\psi$ defined on atoms of the clause parts of* $\mathrm{es}(\varphi)$ *and* $\mathrm{es}(\psi)$*. Then $\chi$ is also a* CERES*-proof.*

*Proof.* CERES-proofs are closed under substitution; so resolving two CERES-proofs reduces to proof resolution. It is easy to see that the linearity condition on the atom parts is preserved under proof resolution. □

**Corollary 8.1.** *Let $\varphi^*$ be an* $\mathrm{ACNF}^{\mathrm{top}}$*-normal form of $\varphi$ and $\chi$ be a proof derived by proof resolution from $\mathcal{P}(\varphi^*)$, the set of projections of $\varphi^*$. Then $\chi$ is weakly intuitionistic.*

*Proof.* All projections are CERES-proofs and, by Proposition 8.1, only CERES-proofs can be derived from them. By construction of the proof resolutions, all end-sequent parts are intuitionistic. □

We have seen in Theorem 5.1 that, for an intuitionistic proof $\psi$ and a proof subsumption $(\varphi, \psi, \eta)$, $\varphi$ is intuitionistic as well; this property is crucial to the completeness theorem. It also generalizes to weakly intuitionistic proofs.

**Theorem 8.2.** *Let $\psi$ be a weakly intuitionistic* CERES*-proof and $(\varphi, \psi, \eta)$ be a proof subsumption. Then $\varphi$ is weakly intuitionistic.*

*Proof.* We can transform every CERES-proof $\psi'$ to a proof $\psi$, where all atoms in the clause part are labeled by numbers (to distinguish them from atoms in the end-sequent part). As there are no inferences on atoms in the clause part, there are also no contractions on these atoms keeping the indexing stable. Therefore, if $S\eta \subseteq S'$, then both $(e(S), e(S'), \eta)$ and $(c(S), c(S'), \eta)$ are proof subsumptions as well. Now, as for every sequent $S$ occurring in $\varphi$, there must be a sequent $S'$ in $\psi$ such that $(S, S', \eta)$ is a proof subsumption ,we also get that $(e(S), e(S'), \eta)$ is a proof subsumption. So, as $\psi$ is weakly intuitionistic, $e(S')$ is intuitionistic and so $e(S)$ is intuitionistic. Hence, $\varphi$ is weakly intuitionistic. □

**Definition 8.3 (i-resolution).** Let $\varphi, \psi$ be CERES-proofs and $\chi$ be a resolvent of $\varphi$ and $\psi$ such that $\chi$ is weakly intuitionistic. Then $\chi$ is called an i-resolvent. A resolution derivation $R$ is called an i-resolution deduction if all proofs in $R$ are weakly intuitionistic. We define CERES-$\mathrm{i}^0$ as the method CERES-i where resolution is replaced by i-resolution.

**Theorem 8.3** (completeness of CERES-$\mathrm{i}^0$)**.** *Let $\varphi$ be an* **LJ***-proof of a skolemized end-sequent $S$. Then the application of* CERES*-$i^0$ to $\varphi$ yields a cut-free* **LJ***-proof $\chi$ of $S$.*

*Proof.* By Theorem 8.1 we know that `CERES-i` is complete. Essentially the refinement used by `CERES-i`$^0$ provides an ordering which we can use to guide the proof resolution. Corollary 8.1 provides a basecase for induction over this ordering. For the step case it is possible for certain proof resolution steps on weakly intuitionistic proofs to result in a proof which is not weakly intuitionistic. But by Lemma 6.1 and Theorem 8.2 we can always fix these cases in order to avoid the loss of weak intuitionism and maintain the property. If we start from the projection of an ACNF$^{\text{top}}$-normal form, we know that the clause set present in those projection will always reach bottom and thus at some point the result of proof resolution will be an **LJ**-proof. □

**Example 8.2.** Let $\varphi$ be the proof

$$\dfrac{\begin{array}{cc}(\varphi_1) & (\varphi_2)\\ \forall x.Qx, \forall x.(Qx \to Px) \vdash \forall x.(P^1x \lor \neg P^2x) & \forall x.(P^1x \lor \neg P^2x) \vdash \neg\neg Pa \to Pa\end{array}}{\forall x.Qx, \forall x.(Qx \to Px) \vdash \neg\neg Pa \to Pa}\ cut$$

where $\varphi_1 =$

$$\dfrac{\dfrac{\dfrac{\dfrac{Qy \vdash Qy \quad Py \vdash P^1y}{Qy, Qy \to Py \vdash P^1y}\ \to_l}{\forall x.Qx, \forall x.(Qx \to Px) \vdash P^1y}\ \forall_l - 2\times}{\forall x.Qx, \forall x.(Qx \to Px) \vdash P^1y \lor \neg P^2y}\ \lor_r}{\forall x.Qx, \forall x.(Qx \to Px) \vdash \forall x.(P^1x \lor \neg P^2x)}\ \forall_r$$

and $\varphi_2 =$

$$\dfrac{\dfrac{\dfrac{\dfrac{Pa^1 \vdash Pa}{Pa^1, \neg\neg Pa \vdash Pa}\ w_l}{Pa^1 \vdash \neg\neg Pa \to Pa}\ \to_r \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Pa \vdash Pa^2}{Pa, \neg Pa^2 \vdash}\ \neg_l}{\neg Pa^2 \vdash \neg Pa}\ \neg_r}{\neg Pa^2, \neg\neg Pa \vdash}\ \neg_l}{\neg Pa^2, \neg\neg Pa \vdash Pa}\ w_r}{\neg Pa^2 \vdash \neg\neg Pa \to Pa}\ \to_r}{Pa^1 \lor \neg P^2a \vdash \neg\neg Pa \to Pa}\ \lor_l}{\forall x.(P^1x \lor \neg P^2x) \vdash \neg\neg Pa \to Pa}\ \forall_l$$

Then our characteristic clause set is $\{\vdash Py^1;\ Pa^1 \vdash;\ \vdash Pa^2\}$. We have three projections $\pi_1, \pi_2, \pi_3$ for

$$\pi_1 = \dfrac{\dfrac{Qy \vdash Qy \quad Py \vdash P^1y}{Qy, Qy \to Py \vdash P^1y}\ \to_l}{\forall x.Qx, \forall x.(Qx \to Px) \vdash P^1y}\ \forall_l - 2\times \qquad \pi_2 = \dfrac{\dfrac{Pa^1 \vdash Pa}{Pa^1, \neg\neg Pa \vdash Pa}\ w_l}{Pa^1 \vdash \neg\neg Pa \to Pa}\ \to_r$$

and

$$\pi_3 = \dfrac{\dfrac{\dfrac{\dfrac{Pa \vdash Pa^2}{\vdash Pa^2, \neg Pa}\ \neg_r}{\neg\neg Pa \vdash Pa^2}\ \neg_l}{\neg\neg Pa \vdash Pa^2, Pa}\ w_r}{\vdash Pa^2, \neg\neg Pa \to Pa}\ \to_r$$

All projections are weakly intuitionistic, $\pi_1$ and $\pi_2$ are intuitionistic - but $\pi_3$ is not. It is easy to see that no proof resolution of $\pi_2$ and $\pi_3$ is either intuitionistic or weakly intuitionistic, for the simple reason that the end-sequent of the resolution is a cut-free proof $\psi$ of $\vdash \neg\neg Pa \to Pa$. As this sequent is not intuitionistically provable, $\psi$ cannot be an intuitionistic proof. Clearly, also none of the resolvents of $\pi_2$ and $\pi_3$ is an i-resolvent. Therefore, we have to resolve $\pi_1$ and $\pi_2$ on $(Py^1, Pa^1)$ with most general unifier $\{y \leftarrow a\}$. One resolvent of $\pi_1$ and $\pi_2$ is

$$
\begin{prooftree}
\AxiomC{$Qa \vdash Qa$}
\AxiomC{$Pa \vdash Pa$}
\RightLabel{$w_l$}
\UnaryInfC{$Pa, \neg\neg Pa \vdash Pa$}
\RightLabel{$\to_r$}
\UnaryInfC{$Pa \vdash \neg\neg Pa \to Pa$}
\RightLabel{$\to_l$}
\BinaryInfC{$Qa, Qa \to Pa \vdash \neg\neg Pa \to Pa$}
\RightLabel{$\forall_l - 2\times$}
\UnaryInfC{$\forall x.Qx, \forall x.(Qx \to Px) \vdash \neg\neg Pa \to Pa$}
\end{prooftree}
$$

which is a result of the method CERES-i$^0$.

## 9. Applications of CERES-i

The method CERES-i and its improvement CERES-i$^0$ form a basic tool for proof analysis in intuitionistic logic. Classical CERES has been used successfully in analyzing nontrivial mathematical proofs like Fürstenberg's proof of the infinitude of primes [3]. As this proof is intrinsically classical it is not the right candidate for an analysis via CERES-i. We rather think about an application to a mathematical proof in linear algebra. In particular, we plan to formalize a proof that a particular system of linear equations has a solution, where the proof uses the Kronecker-Capelli theorem [18](the theorem says that, given an $n \times n$ matrix $A$ and an $n$-vector $\vec{b}$, the system $A\vec{x} = \vec{b}$ is solvable iff $\mathit{rank}(A) = \mathit{rank}(A|b)$). Thus one can prove the solvability of $A\vec{x} = \vec{b}$ by proving first $\mathit{rank}(A) = \mathit{rank}(A|b)$ and using cut with the Kronecker-Capelli theorem. Then by applying cut-elimination one can obtain a concrete solution for $A\vec{x} = \vec{b}$. In the case of a natural deduction formulation and normalization, this would yield exactly one solution even if there are infinitely many. By the non-confluence inherent in the CERES (and also the CERES-i) method, we can expect to obtain different solutions even in the intuitionistic case. Note that the application of the *classical* CERES method would produce a *set of candidate solutions*, where the correct solution still needs to be identified after application of cut-elimination. Hence by application of different methods of cut-elimination to a single intuitionistic proof, we can expect different kinds of results. We intend to systematically compare their strengths and weaknesses in the context of this case study.

At this point, there is no implementation of the method CERES-i, but we plan to implement it based on *indexed resolution*[6] as a search heuristic for resolution refutations.

In order to use indexed resolution, we need to assign indices to the atoms occurring in the cut-formulas of a proof. The assignment of these indices is done in such a way that in each cut the two opposite occurrences of the cut-formula

[6]See Section 7 for indexed resolution restricted to proofs in ACNF$^{\text{top}}$.

have exactly the same indices and each atom within the cut-formula gets a different index. As a consequence, all atomic cut-ancestors in such an indexed proof have an assigned index as well. Additionally, we require that no two cut-formulas coming from different cuts have the same indexing. This situation may change after applying cut-reduction steps, but the initial assignment has to fulfil this requirement. For an example of a proof with atom indexing, see the proof $\varphi'$ below. The indexed resolution rule itself is then just a restricted form of the ordinary resolution rule, where we only allow to resolve upon atoms having the same index.

Indexed resolution has the nice feature that it drastically reduces the search space for resolution refutations of the characteristic clause set. This is due to the fact that many clauses that could have been resolved using ordinary resolution can no longer be resolved after taking their assigned indices into account.

We conjecture that *indexed* resolution is complete for `CERES`-i in the following sense: every *indexed* resolution refutation of the characteristic clause set $\mathrm{CL}(\varphi)$ of a proof $\varphi$ can serve as a skeleton for a `CERES`-i normal form. Using this refinement we can reduce the search space considerably. Note that the method `CERES`-i with ordinary resolution applied to an intuitionistic proof does not have this nice property, as the proof $\varphi$ of $P \vee \neg P \vdash \neg\neg P \to P$ in Section 3.1 demonstrates.

However, in the case of indexed resolution we can circumvent this problem using the method `CERES`-i. By assigning to each atomic cut-ancestor in $\varphi$ a unique index, we can obtain from $\varphi$ an indexed proof $\varphi'$ (with the same end-sequent) of the following form:

$$
\begin{prooftree}
\AxiomC{$P \vdash P^1$}
\RightLabel{$\vee_r$}
\UnaryInfC{$P \vdash P^1 \vee \neg P^2$}
\AxiomC{$P^2 \vdash P$}
\RightLabel{$\neg_l$}
\UnaryInfC{$P^2, \neg P \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg P \vdash \neg P^2$}
\RightLabel{$\vee_r$}
\UnaryInfC{$\neg P \vdash P^1 \vee \neg P^2$}
\RightLabel{$\vee_l$}
\BinaryInfC{$P \vee \neg P \vdash P^1 \vee \neg P^2$}
\AxiomC{$P \vdash P^2$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg P^2, P \vdash$}
\RightLabel{$\neg_r$}
\UnaryInfC{$\neg P^2 \vdash \neg P$}
\RightLabel{$\neg_l$}
\UnaryInfC{$\neg P^2, \neg\neg P \vdash$}
\RightLabel{$w_r$}
\UnaryInfC{$\neg P^2, \neg\neg P \vdash P$}
\RightLabel{$\to_r$}
\UnaryInfC{$\neg P^2 \vdash \neg\neg P \to P$}
\AxiomC{$P^1 \vdash P$}
\RightLabel{$w_l$}
\UnaryInfC{$P^1, \neg\neg P \vdash P$}
\RightLabel{$\to_r$}
\UnaryInfC{$P^1 \vdash \neg\neg P \to P$}
\RightLabel{$\vee_l$}
\BinaryInfC{$P^1 \vee \neg P^2 \vdash \neg\neg P \to P$}
\RightLabel{$cut$}
\BinaryInfC{$P \vee \neg P \vdash \neg\neg P \to P$}
\end{prooftree}
$$

The characteristic clause set $\mathrm{CL}(\varphi')$ is the following

$$\mathrm{CL}(\varphi') = \{P^2 \vdash P^1 \ ; \ P^1 \vdash ; \ \vdash P^2\},$$

First, consider the indexed resolution refutation of $\mathrm{CL}(\varphi')$:

$$
\begin{prooftree}
\AxiomC{$\vdash P^2$}
\AxiomC{$P^2 \vdash P^1$}
\AxiomC{$P^1 \vdash$}
\RightLabel{$R$}
\BinaryInfC{$P^2 \vdash$}
\RightLabel{$R$}
\BinaryInfC{$\vdash$}
\end{prooftree}
$$

The corresponding minimal projections $\varphi'[\vdash P^2]$, $\varphi'[P^1 \vdash]$, and $\varphi'[P^2 \vdash P^1]$

are

$$
\dfrac{\dfrac{\dfrac{\dfrac{P \vdash P^2}{\vdash P^2, \neg P}\,\neg_r}{\neg\neg P \vdash P^2}\,\neg_l}{\neg\neg P \vdash P^2, P}\,w_r}{\vdash P^2, \neg\neg P \to P}\,\to_r
\qquad
\dfrac{\dfrac{P^1 \vdash P}{P^1, \neg\neg P \vdash P}\,w_l}{P^1 \vdash \neg\neg P \to P}\,\to_r
\qquad
\dfrac{P \vdash P^1 \quad \dfrac{\dfrac{P^2 \vdash P}{\neg P, P^2 \vdash}\,\neg_r}{\neg P, P^2 \vdash P^1}\,w_r}{P \vee \neg P, P^2 \vdash P^1}\,\vee_l
$$

Based on these indexed projections, we end up with the following proof:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P \vdash P^2}{\vdash P^2, \neg P}\,\neg_r}{\neg\neg P \vdash P^2}\,\neg_l}{\neg\neg P \vdash P^2, P}\,w_r}{\vdash P^2, \neg\neg P \to P}\,\to_r
\quad
\dfrac{\dfrac{P \vdash P^1 \quad \dfrac{\dfrac{P^2 \vdash P}{\neg P, P^2 \vdash}\,\neg_r}{\neg P, P^2 \vdash P^1}\,w_r}{P \vee \neg P, P^2 \vdash P^1}\,\vee_l
\quad
\dfrac{\dfrac{P^1 \vdash P}{P^1, \neg\neg P \vdash P}\,w_l}{P^1 \vdash \neg\neg P \to P}\,\to_r}{P \vee \neg P, P^2 \vdash \neg\neg P \to P}\,cut}{P \vee \neg P \vdash \neg\neg P \to P, \neg\neg P \to P}\,cut}{P \vee \neg P \vdash \neg\neg P \to P}\,c_r
$$

The above proof still violates the restrictions of **LJ**, but it does not contain a proof of an intuitionistically invalid sequent anymore. In order to get a `CERES`-i normal form we need to apply (ground) proof resolution to the projections of $\mathrm{CL}(\varphi')$.

Proof resolution on $\varphi'[P^2 \vdash P^1]$ and $\varphi'[P^1 \vdash]$ yields the following result $\psi$:

$$
\dfrac{\dfrac{\dfrac{P \vdash P \quad \dfrac{\dfrac{P^2 \vdash P}{\neg P, P^2 \vdash}\,\neg_r}{\neg P, P^2 \vdash P}\,w_r}{P \vee \neg P, P^2 \vdash P}\,\vee_l}{P \vee \neg P, P^2, \neg\neg P \vdash P}\,w_l}{P \vee \neg P, P^2 \vdash \neg\neg P \to P}\,\to_r
$$

After applying proof resolution on $\psi$ and $\varphi'[\vdash P^2]$ we arrive at the following `CERES`-i normal form $\psi'$:

$$
\dfrac{\dfrac{P \vdash P \quad \dfrac{\dfrac{\dfrac{\dfrac{P \vdash P}{\neg P, P \vdash}\,\neg_l}{\neg P \vdash \neg P}\,\neg_r}{\neg P, \neg\neg P \vdash}\,\neg_l}{\neg P, \neg\neg P \vdash P}\,w_r}{P \vee \neg P, \neg\neg P \vdash P}\,\vee_l}{P \vee \neg P \vdash \neg\neg P \to P}\,\to_r
$$

Clearly, $\psi'$ is an intuitionistic proof of $P \vee \neg P \vdash \neg\neg P \to P$.

## 10. Complexity results

The elimination of cuts in a sequent calculus proof is, in general, of non-elementary complexity. The lower bound on this complexity was established by Statman in [19] (later the upper bound was shown to be the same [11]) and the same proof sequence can be used to show that the *classical* `CERES` method is also non-elementary [7, Section 6.5]. The main source of complexity in cut-elimination by resolution is the size of resolution refutations. Since the intuitionistic method `CERES`-i also relies on the refutation of the clause set, we can safely say that a non-elementary function is also its lower bound.

Nevertheless, the `CERES` method (for classical first-order logic) was shown to give a non-elementary speed up over two reductive methods: one that reduces always an uppermost cut (Gentzen) and another that reduces always a most complex cut (Tait-Schütte). That is, there exist sequences of proofs for which the size of the largest proof constructed during those reductive cut-elimination methods is non-elementary on the size of the initial proof, though, for the exact same sequence, there exists elementary-sized refutations of the sequence of characteristic clause sets [7, Section 6.10]. Moreover it is shown that neither the Gentzen nor the Tait-Schütte method (provided they construct an ACNF and do not use the axiom rule) can speed up `CERES` non-elementarily [7, Theorem 6.10.3]. Without restricting the cut-elimination strategy, it was shown that `CERES` can be exponentially worse than reductive methods [20].

In this section, we compare the `CERES`-i method with specific reductive cut-elimination strategies to obtain similar results. Our complexity analysis is based on the space complexity of the methods, measuring the objects of maximal size constructed by the algorithms. For the first result, we use the reductive cut-elimination that first obtains an $\mathrm{ACNF}^{\mathrm{top}}$ from the original proof with cuts and only then removes the atomic cuts. We show that the space complexity of `CERES`-i is an elementary function on the size of the $\mathrm{ACNF}^{\mathrm{top}}$. Since this size is a lower bound on the space complexity of the reductive cut-elimination considered, we get as a result that the complexity of `CERES`-i is elementarily bound in that of this reductive method. For the second result, we use the aforementioned Gentzen method, which reduces always the uppermost cut first. We show that there is a sequence of proofs for which the space complexity of `CERES`-i is constant, while reductive cut-elimination using the Gentzen method is non-elementary. By taking the intersection of both reductive methods (i.e. obtaining the $\mathrm{ACNF}^{\mathrm{top}}$ using the Gentzen strategy and then removing the atomic cuts), we have a reductive cut-elimination method which is outperformed by `CERES`-i.

**Definition 10.1.** We define the *size* of a sequent calculus proof $\varphi$, denoted by $|\varphi|$ as the number of symbols used in the proof (including logical connectives, parenthesis, meta-connectives such as the sequent sign, as well as all the symbols used for predicates and terms).

In what follows we show the relation between the sizes of the $\mathrm{ACNF}^{\mathrm{top}}$ and the cut-free proof obtained by `CERES`-i.

**Lemma 10.1.** *Let $\varphi$ be an* **LJ***-proof with cuts, $\varphi_0^t$ its* $\mathrm{ACNF}^{\mathrm{top}}$*, and $P_{max}$ the largest projection in $\mathcal{P}(\varphi_0^t)$. Let $\psi$ be a proof derived by proof resolution of the projections $\mathcal{P}(\varphi_0^t)$, and let $r$ be the number of resolution steps. Then $|\psi| \leq 2^r \times |P_{max}|$.*

*Proof.* Consider two proofs $\chi_1$ and $\chi_2$ that are resolved (via proof resolution) into $\chi$. Then it is safe to say that $|\chi| \leq |\chi_1| + |\chi_2| \leq 2 \times max(|\chi_1|, |\chi_2|)$. Now, suppose that $\chi$ is the result of the first resolution step of the projections $\mathcal{P}(\varphi_0^t)$. Then $|\chi| \leq 2 \times |P_{max}|$. All following resolution steps will, at most, double this value. Since there are $r$ resolution steps, we can conclude that $|\psi| \leq 2^r \times |P_{max}|$. □

**Theorem 10.1.** *Let $\varphi$ be an* **LJ***-proof with cuts, $\varphi^t$ its* ACNF$^{\text{top}}$ *and $\psi$ a cut-free proof derived by a proof resolution of the projections $\mathcal{P}(\varphi^t)$ in $r$ resolution steps Then $|\psi| \leq 2^r \times |\varphi^t|$.*

*Proof.* From Lemma 10.1, we know that $|\psi| \leq 2^r \times |P_{max}|$, where $P_{max}$ is the biggest projection in $\mathcal{P}(\varphi^t)$. Given the definition of projection, we can assume that $|P_{max}| \leq |\varphi^t|$. Using these two inequalities we obtain the desired result: $|\psi| \leq 2^r \times |\varphi^t|$ □

Observe that this result uses the normal form obtained by `CERES`-i applied to a proof in ACNF$^{\text{top}}$. In order to generalize it for any kind of **LJ** proof, we will use the subsumption property.

**Lemma 10.2.** *Let $\varphi$ be an* **LJ***-proof with cuts and $\varphi^t$ its* ACNF$^{\text{top}}$*. Let $\psi$ be a normal form obtained by applying* `CERES`*-i to $\varphi$ and $\psi'$ be a normal form obtained by applying* `CERES`*-i to $\varphi^t$. Then $|\psi| \leq |\psi'|^k$ for some $k \in \mathbb{R}$.*

*Proof.* By Theorem 7.2, there exists a proof resolution of the projections $\mathcal{P}(\varphi^t)$ yielding a cut-free **LJ**-proof $\psi$. By Theorem 5.2, all projections in $\mathcal{P}(\varphi^t)$ are subsumed by the projections in $\mathcal{P}(\varphi)$. By Lemma 6.1, there exists a proof resolution of $\mathcal{P}(\varphi)$ resulting in a proof $\psi'$ that subsumes $\psi$, the proof resolution of $\mathcal{P}(\varphi^t)$. Now that we clarified this relationship we can proceed with bounding the size $\psi$ in terms of $\psi'$. This does not generally hold for subsuming proofs, but holds in this case, since the proof resolutions are derived from the same proof. We prove this by induction on the subsumption (Definition 5.3).

**Base case:** For the base case we have that $\psi$ is an axiom and that $es(\psi)\sigma \subseteq es(\psi')$. Clearly, $|\psi| \leq |\psi'|$. [7]

**Inductive cases:**

The inequality holds for all inductive cases trivially, except 3(b)i, 3(b)ii, 4a and 4b. The cases 3(b)i and 3(b)ii are analogous, as are cases 4a and 4b, therefore, we will restrict the analysis to one of each.

- Case 3(b)i: The proofs $\psi$ and $\psi'$ are, respectively:

$$\dfrac{\dfrac{\begin{matrix}(\psi_1)\\ A_0, \Gamma' \vdash \Delta\end{matrix}}{A_0, \Gamma' \vdash \Delta, B}\, w_r}{\Gamma' \vdash \Delta, A_0 \to B}\, \to_r \qquad \dfrac{\begin{matrix}(\psi_1')\\ A, \Pi \vdash \Lambda, B\end{matrix}}{\Pi \vdash \Lambda, A \to B}\, \to_r$$

[7] Observe that if we had not restricted ourselves to `CERES`-i normal forms, we could have a situation where $\psi'$ is an axiom but $\psi$ is not (imagine a proof that has an axiom as its end-sequent, and only nonsensical structural rules above it).

By inductive hypothesis we assume that $|\psi_1| \leq |\psi'_1|^k$. We know that $(\Gamma' \vdash \Delta, A_0 \rightarrow B)\sigma \subseteq \Pi \vdash \Lambda, A \rightarrow B$, therefore the inequality still holds by adding the symbols of both end-sequents to the respective proofs. We need to worry only with the sequent $A_0, \Gamma' \vdash \Delta, B$ in $\varphi$. But note that the number of symbols in this sequent is strictly less than the number of symbols in $es(\psi')$. We can thus safely say that $|\psi| \leq 2 \times |\psi'| \leq |\psi'|^k$ for a sufficiently large $k$.

- Case 4a: The proofs $\psi$ and $\psi'$ are, respectively:

$$\dfrac{\begin{matrix}(\psi_1)\\ \Gamma_1 \vdash \Delta_1\end{matrix}}{\Gamma_1, \Gamma^* \vdash \Delta_1, \Delta^*}\; w_l^*, w_r^* \qquad \dfrac{\begin{matrix}(\psi'_1)\\ \Pi_1 \vdash \Lambda_1, A\end{matrix} \quad \begin{matrix}(\psi'_2)\\ \Pi_2 \vdash \Lambda_2, B\end{matrix}}{\Pi_1, \Pi_2 \vdash \Lambda_1, \Lambda_2, A \wedge B}\; \wedge_r$$

By inductive hypothesis we assume that $|\psi_1| \leq |\psi'_1|^c$. We know that $(\Gamma_1, \Gamma^* \vdash \Delta_1, \Delta^*)\sigma \subseteq (\Pi_1, \Pi_2 \vdash \Lambda_1, \Lambda_2)$, therefore the end-sequents do not affect the inequality. We need only to compare the number of symbols in the sequence of weakenings of $\Gamma^*, \Delta^*$ (in $\psi$) with those in $\psi'_2$ (in $\psi'$). Note that $\Gamma^* \subseteq \Pi_2$ and $\Delta^* \subseteq \Lambda_2$, so the symbols introduced by $\Gamma^*, \Delta^*$ are at most the size of $\psi'_2$ (worst case scenario, these formulas were also weakened in $\psi'_2$, and it contains an additional $B$). We need only to worry about the copies of $\Gamma_1, \Delta_1$. Note that at most $|\Gamma^*| + |\Delta^*| \leq |\psi'_2| \leq |\psi'|$ number of weakenings were performed, copying at most $|\Gamma_1| + |\Delta_1| \leq |\psi'_1| \leq |\psi'|$ formulas. Then, $|\psi| \leq |\psi'_1|^c + |\psi'|^2 \leq |\psi'|^k$ for a sufficiently large $k$.

□

**Theorem 10.2.** *Let $\varphi$ be an **LJ**-proof with cuts, $\varphi^t$ its ACNF$^{\text{top}}$, $\psi'$ be a cut-free proof derived by a proof resolution of the projections $\mathcal{P}(\varphi^t)$ in $s$ resolution steps, and $\psi$ be a cut-free proof derived by a proof resolution of the projections $\mathcal{P}(\varphi)$. Then $|\psi| \leq 2^{sk} \times |\varphi^t|^k$, for some constant $k$, which is independent of $\varphi$ and $\psi$.*

*Proof.* By Theorem 10.1, we know that $|\psi'| \leq 2^s \times |\varphi^t|$. By Lemma 10.2 we know that $|\psi| \leq |\psi'|^k$ for some constant $k$. Thus, $|\psi| \leq |\psi'|^k \leq (2^s \times |\varphi^t|)^k$ Using these results, we can conclude that $|\psi| \leq 2^{sk} \times |\varphi^t|^k$. □

We have shown that the size of the final proof using `CERES`-i is an exponential function on the number of resolution steps and a polynomial function on the size of the proof obtained by the reductive cut-elimination method considered. We now use this result to obtain a relation between the space complexity of the methods.

**Definition 10.2 ($\mathcal{R}$ cut-elimination sequence).** Let $\mathcal{R}$ be a set of cut reduction rules and $\Phi$ be the cut-elimination sequence: $\varphi_0 \rightsquigarrow_{\mathcal{R}} \varphi_1 ... \rightsquigarrow_{\mathcal{R}} \varphi_n$ where $\varphi_n$ is cut-free. Then $\Phi$ is called a $\mathcal{R}$ *cut-elimination sequence* on $\varphi_0$. We define the *size* of $\Phi$ as $|\Phi| = \max\{|\varphi_i| \mid 0 \leq i \leq n\}$.

The complexity of a reductive cut-elimination method is given by the smallest possible cut-elimination sequence on a given proof.

**Definition 10.3 (Complexity of reductive cut-elimination).** Let $\mathcal{R}$ be a system of reductive cut-elimination and $\varphi$ be an **LJ**-proof. Then the complexity of reductive cut-elimination on $\varphi$ via $\mathcal{R}$ is defined as

$$\|\varphi\|_{\mathcal{R}} = \min\{|\Phi| \mid \Phi \text{ is a cut-elimination sequence on } \varphi\}.$$

**Definition 10.4.** Let $\mathcal{R}^{top}$ denote the reduction strategy for cut elimination that first obtains an $\mathrm{ACNF}^{\mathrm{top}}$ using the rewrite rules $\mathcal{R}^a$ and only then removes the atomic cuts. Every $\mathcal{R}^{top}$ cut-elimination sequence on a proof $\varphi_0$ is of the form:

$$\varphi_0 \rightsquigarrow_{\mathcal{R}^a} \ldots \varphi_k \rightsquigarrow_{\mathcal{R}} \ldots \varphi_n$$

where $\varphi_k$ is an $\mathrm{ACNF}^{\mathrm{top}}$ of $\varphi_0$ and $\varphi_n$ is cut-free.

**Theorem 10.3.** *There exists an elementary function $f$ such that, given an* **LJ**-*proof $\varphi$ and an $\mathcal{R}^{top}$ cut-elimination sequence $\Phi$ of the form $\varphi \rightsquigarrow^*_{\mathcal{R}^a} \varphi^t \rightsquigarrow^*_{\mathcal{R}} \varphi^0$ (where $\varphi^t$ is an* $\mathrm{ACNF}^{\mathrm{top}}$*), $|\varphi^t| \leq f(|\Phi|)$.*

*Proof.* In the general case, the final proof in a cut-elimination reduction sequence might be non-elementarily bigger than the initial proof with cuts. But in the case of this strategy, the final transformation from $\varphi^t$ to a cut free $\varphi'$ only reduces the size of the proof. Therefore, a potential "non-elementarity" must already occur at the $\mathrm{ACNF}^{\mathrm{top}}$. $\square$

The method `CERES`-i can also be described via a sequence of proofs beginning with a proof $\varphi_0$ and ending with a cut free proof $\varphi_n$. However, the difference to reductive cut-elimination is that the occurring proofs are either projections of the original proof $\varphi$ or resolvents of proofs occurring earlier in the sequence.

**Definition 10.5 (`CERES`-i sequence).** Let $\varphi$ be an **LJ**-proof and and $\mathcal{P}(\varphi)$ its set of projections. A sequence $\Psi\colon \varphi_0, \ldots, \varphi_n$ is called a `CERES`-i sequence if $\varphi_0 \in \mathcal{P}(\varphi)$ and for all $i \in \{1, \ldots, n\}$ either $\varphi_i \in \mathcal{P}(\varphi)$ or there are $j, k < i$ such that $\varphi_i$ is a proof resolvent of $\varphi_j$ and $\varphi_k$. If $\varphi_n$ is cut free and in **LJ** then we call $\Psi$ a `CERES`-i cut-elimination sequence on $\varphi$. The *size* of $\Psi$ is defined as $|\Psi| = \max\{|\varphi_i| \mid 0 \leq i \leq n\}$.

**Definition 10.6 (Complexity of `CERES`-i).** Let $\varphi$ be an **LJ**-proof. Then the `CERES`-i cut-elimination complexity of $\varphi$ is defined as

$$\|\varphi\|_{\mathtt{C}} = \min\{|\Psi| \mid \Psi \text{ is a } \mathtt{CERES}\text{-i cut-elimination sequence on } \varphi\}.$$

**Theorem 10.4.** *Let $\varphi$ be an* **LJ**-*proof and $\psi$ be a cut free proof of minimal size obtained by applying* `CERES`-*i. Then $\|\varphi\|_{\mathtt{C}} = |\psi|$.*

*Proof.* Let $\chi$ be the result of resolving two proofs $\chi_1$ and $\chi_2$. Then it is the case that $|\chi| \geq max(|\chi_1|, |\chi_2|)$. Since the result of proof resolution is always bigger than its operands, we can conclude that the last proof in the proof resolution steps for `CERES`-i *is* the biggest one and therefore defines the complexity of the sequence. □

Now we can obtain the desired result.

**Theorem 10.5.** *There exists an elementary function $g$ such that for all* **LJ**-*proofs $\varphi$ we have*

$$\|\varphi\|_{\mathtt{C}} \leq g(\|\varphi\|_{\mathcal{R}^{top}})$$

*Proof.* Let $\varphi^t$ be the $\mathrm{ACNF}^{\mathrm{top}}$ obtained during reductive cut-elimination using $\mathcal{R}^{top}$ and $\psi$ the cut free proof obtained by `CERES`-i. Then, by Theorem 10.2 we know that $|\psi| \leq 2^k \times |\varphi^t|^{k'}$ for some $k$ and $k'$. Using the results of Theorems 10.3 and 10.4 on this inequality we obtain $\|\varphi\|_{\mathtt{C}} \leq 2^k \times (f(\|\varphi\|_{\mathcal{R}^{top}}))^{k'}$. Since all operations are elementary, we can say that there exists an elementary function $g$ such that $\|\varphi\|_{\mathtt{C}} \leq g(\|\varphi\|_{\mathcal{R}^{top}})$. □

We have thus shown that the space complexity of `CERES`-i is elementarily bounded by the space complexity of the cut elimination method $\mathcal{R}^{top}$. Next we use a different reduction strategy to show that there exists a sequence of proof for which `CERES`-i is constant and the reduction strategy is non-elementary.

**Definition 10.7.** Let $\mathcal{R}^G$ denote the Gentzen reduction strategy for cut elimination that allows rules in $\mathcal{R}$ to be applied only to uppermost cuts.

**Theorem 10.6.** *There exists an infinite sequence of* **LJ**-*proofs $\varphi_n$ of sequents $S_n$ such that*

(a) $\mathcal{R}^G$ is non-elementary*: For all $\mathcal{R}^G$ cut-elimination sequences on $\varphi_n$ and for all elementary functions $g : \mathbb{N} \to \mathbb{N}$, there exists a $k$ such that for every $n > k$ we have $\|\varphi_n\|_{\mathcal{R}^G} > g(|\varphi_n|)$.*

(b) `CERES`-i is constant*: There exists a resolution derivation $\delta$ in* `CERES`-*i of a cut free proof $\psi$ from the projections of $\varphi_n$ such that for all $n \in \mathbb{N}$, $\psi$ subsumes a cut free proof of $S_n$.*

*Proof.* Let $\gamma_n$ be the short proofs with cuts of the Statman-sequence $\Delta_n \vdash D_n$ (for a formal definition of $\gamma_n$ see [4]). Note that $\gamma_n$ is a sequence of intuitionistic proofs. We define $\varphi_n$ as

$$\dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \overset{(\gamma_n)}{\Delta_n \vdash D_n}}{A, \Delta_n \vdash A \wedge D_n}\ \wedge_r \quad \dfrac{\dfrac{}{A \vdash A}}{A \wedge D_n \vdash A}\ \wedge_l}{A, \Delta_n \vdash A}\ cut$$

where $A$ is an atom. We get $\mathrm{CL}(\varphi_n) = \{\vdash A\} \cup \mathrm{CL}(\varphi_n) \cup \{A \vdash\}$. Now consider the minimal projections

$$\varphi[\vdash A] = A \vdash A^*,\ \varphi[A \vdash] = A^* \vdash A.$$

We indicate the cut-ancestors by $*$. Then, obviously $\delta\colon A \vdash A$ is a resolvent of $\varphi[\vdash A]$ and $\varphi[A \vdash]$. $\delta$ subsumes the cut free intuitionistic proof $\chi_n$:

$$\frac{A \vdash A}{A, \Delta_n \vdash A}\ w_l^*$$

This proves b.

a: $\mathcal{R}^G$ is of nonelementary complexity as, by the Gentzen refinement, all cuts in $\gamma_n$ must be eliminated first resulting in a cut free proof $\gamma_n^*$ of $\Delta_n \vdash D_n$. Let $\gamma_n^*$ be the smallest cut free proof of $\gamma_n$; then, clearly, $\|\varphi_n\|_{\mathcal{R}^G} \geq |\gamma_n^*|$. As $\gamma_n$ is the Statman sequence we get

$$\|\varphi_n\|_{\mathcal{R}^G} \geq |\gamma_n^*| > s(n)/2 \text{ for } s(1) = 2, s(n+1) = 2^{s(n)},$$

and for any elementary function $g$ there exist only finitely many $n$ such that $s(n)/2 \leq g(n)$. □

**Corollary 10.1.** *There exists no elementary function $g$ such that for all intuitionistic proofs $\varphi$: $\|\varphi\|_{\mathcal{R}^G} \leq g(\|\varphi\|_{\mathtt{C}})$.*

*Proof.* Immediate by Theorem 10.6. □

When we combine $\mathcal{R}^G$ and $\mathcal{R}^{top}$ we obtain a direct comparison of CERES-i and reductive cut-elimination in both directions. Let $\mathcal{R}^{Gtop}$ be the following cut-elimination method. Given an **LJ**-proof $\varphi$ we proceed as follows:

1. apply $\mathcal{R}^G$ to uppermost non-atomic cuts until only atomic cuts are left and obtain $\varphi^*$.
2. apply $\mathcal{R}^{top}$ to $\varphi^*$ untill a cut free proof is obtained.

Obviously $\mathcal{R}^{Gtop}$ is a refinement of $\mathcal{R}^{top}$. Hence we obtain the result:

**Theorem 10.7.** *It holds that:*

(a) *There exists an elementary function $g$ s.t. for all* **LJ***-proofs $\varphi$: $\|\varphi\|_{\mathtt{C}} \leq g(\|\varphi\|_{\mathcal{R}^{Gtop}})$.*
(b) *There exists no elementary function $g$ s.t. for all intuitionistic proofs $\varphi$: $\|\varphi\|_{\mathcal{R}^{Gtop}} \leq g(\|\varphi\|_{\mathtt{C}})$.*

*Proof.* For a: Theorem 10.5 holds also for $\mathcal{R}^{Gtop}$.

For b: Take the sequence $\varphi_n$ from Theorem 10.6. We have to argue differently as we must first compute an $\text{ACNF}^{\text{top}}$ before eliminating the atomic cuts. Therefore we eliminate all nonatomic cuts in the proof $\gamma_n$ first and obtain an atomic cut normal form $\gamma_n^+$ (all cuts in $\gamma_n^+$ are atomic). As cut-elimination of atomic cuts is (only) exponential there is no elementary bound on $|\gamma_n^+|$ in terms of $|\gamma_n|$ (otherwise there would be an elementary bound on $|\gamma_n^*|$ in terms of $|\gamma_n|$ (see the proof of Theorem 10.6) which does not exist. □

## 11. Conclusion

This paper is the concluding step on our journey to develop a `CERES`-like method for intuitionistic logic. We had perceived from early on that the operations of `CERES` were intrinsically classical, and after many attempts to preprocess or post-process the proofs, we realized that the core of the method needed to be changed. We could no longer separate projections from resolution proofs in intuitionistic logic; hence we introduced a proof resolution principle, which generalizes resolutions on clauses to resolutions on proofs. Thus by resolving projections one can obtain an intuitionistic cut free proof (even if the projections themselves are not intuitionistic due to atomic cut ancestors on the right). The work developed here is dedicated mainly to proving that such method of proof resolution of projections is complete, i.e., resolving projections from an *intuitionistic* proof results eventually in a cut free *intuitionistic* proof. The completeness proof relies on another concept borrowed from clauses and resolutions, namely, proof subsumption.

We believe that one of the main contributions of this work is the definition of these two concepts, which can easily be lifted to classical logic and used in other areas of proof theory.

In addition to the completeness results, we have also shown that `CERES`-i can outperform a particular kind of reductive cut-elimination, namely, that which first obtains an $\mathrm{ACNF}^{\mathrm{top}}$ by reducing the uppermost nonatomic cuts first and, in a second step, removes the atomic cuts.

`CERES`-type methods have been developed for higher-order logic [12], for finitely valued logics [6] and for Gödel logic [1]. For these logics `CERES` turned out to be largely independent of the definition of the chosen calculus. In fact `CERES` can be considered as a *semi-semantic* method compared to the purely syntactic Gentzen type methods of cut-elimination (see [8]). In contrast, `CERES`-i is a purely syntactic method depending on the single-conclusion calculus **LJ** and there is no obvious way to extend this method to a multi-conclusion calculus for intuitionistic logic. In classical logic the completeness of the `CERES`-method can be established by the semantic completeness of resolution in classical logic and *any* resolution refutation can be turned into a atomic cut normal form and finally into a cut-free proof. We have shown that such an approach fails for **LJ**. For **LJ** we developed the method of proof resolution; for proving completeness we used reductive cut-elimination and proof subsumption (instead of Kripke semantics). An investigation of cut-elimination in intermediate logics (like this of Gödel logic baed on a hypersequent calculus in [1]) via `CERES`-like proof resolution methods remains an interesting topic for future research.

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